Verified encodings for SAT solvers

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Advised by Marijn J. H. Heule and Jeremy Avigad

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Repo at https://github.com/ccodel/verified-encodings

The problem with SAT encodings

The Lean theorem prover

Verified encodings library

Applications

SAT solvers are great!

Hardware/software verification, optimization, SMT solvers, ...

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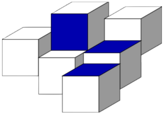
Keller's Conjecture [IJCAR'20]



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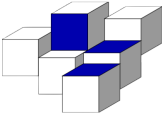

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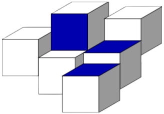

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Lam's Problem [AAAI'21]

1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
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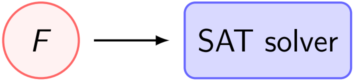
The SAT toolchain



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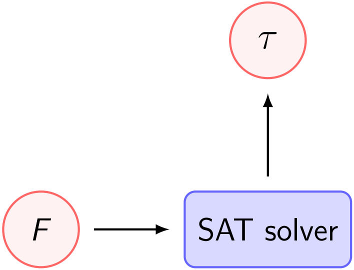

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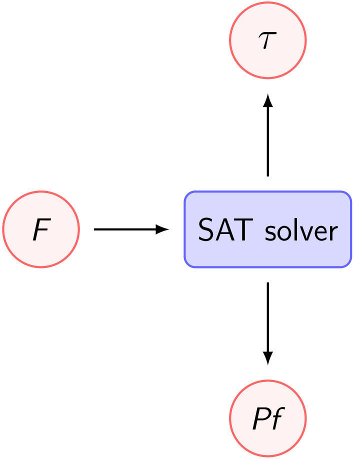

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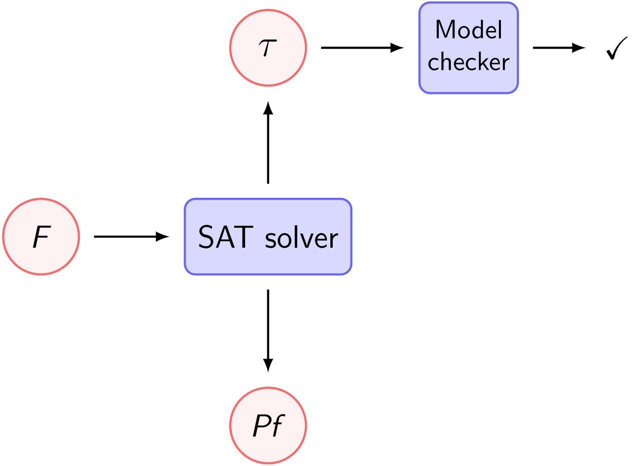

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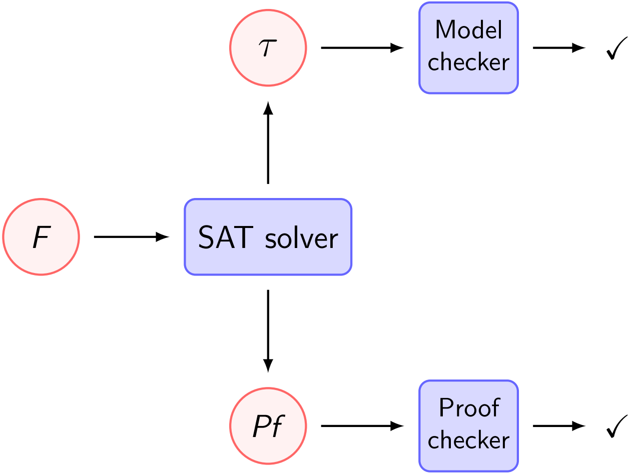

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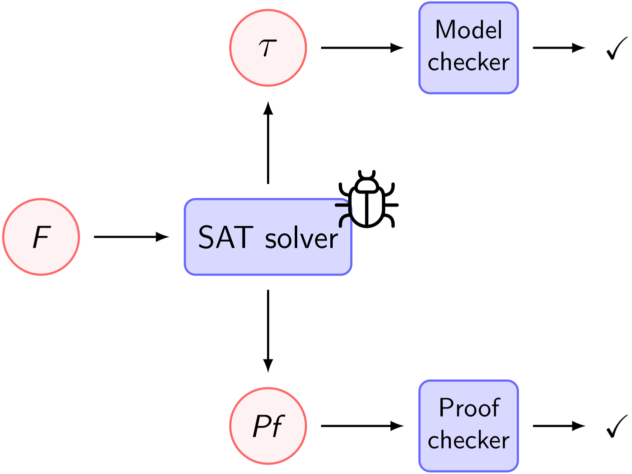

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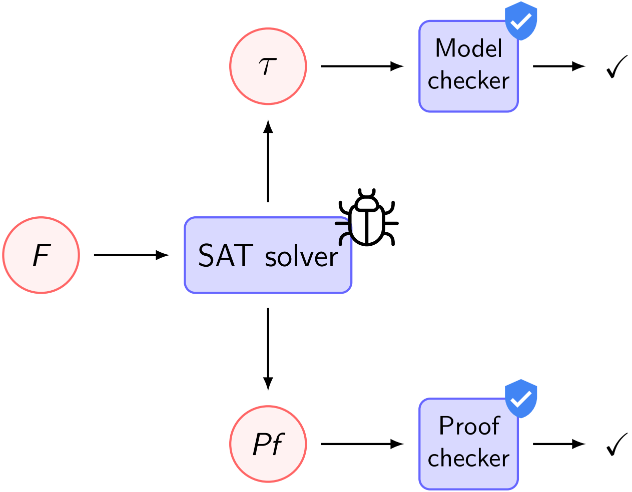

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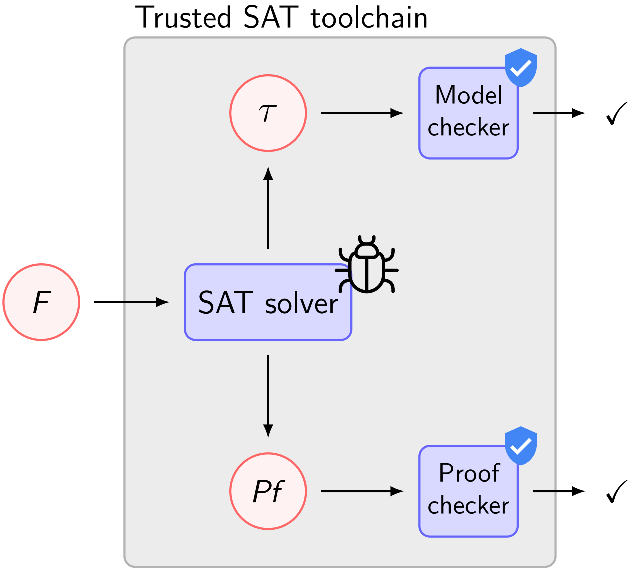

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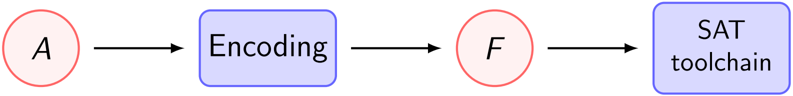

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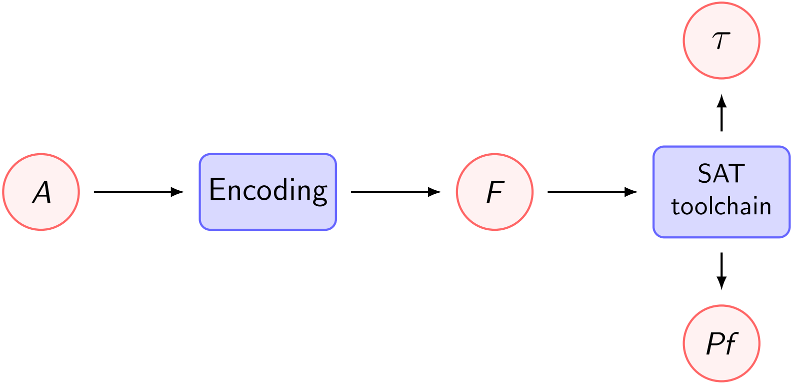

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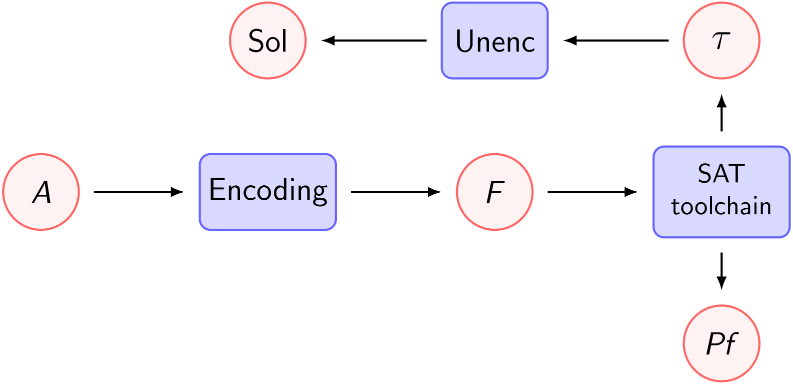

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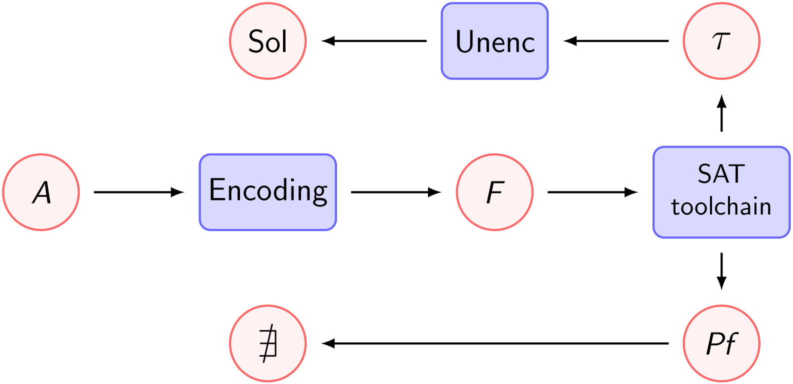

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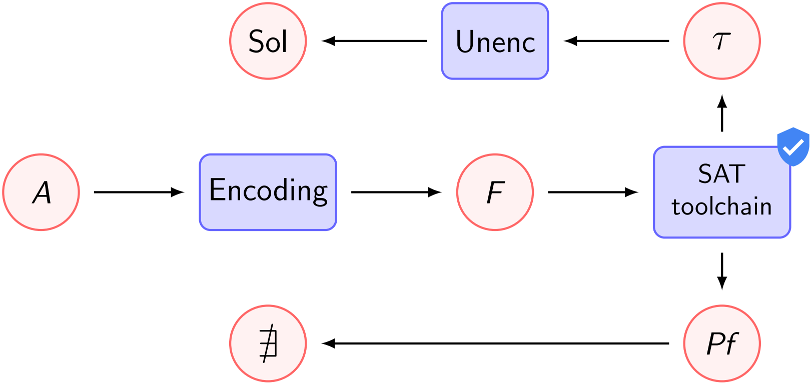

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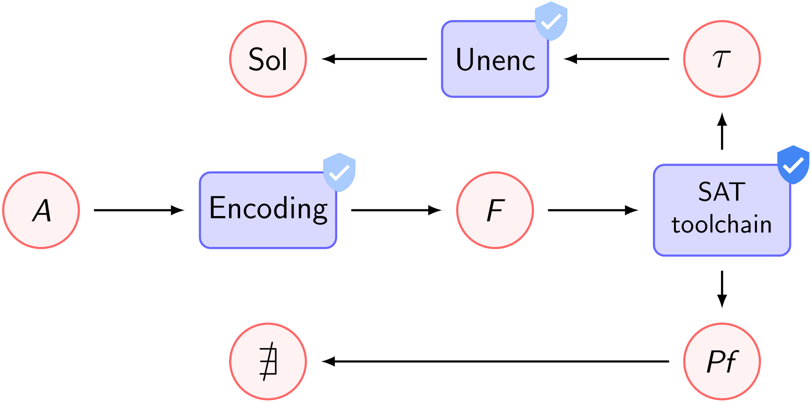

My work: extend the trusted SAT toolchain to include encodings by using a theorem prover

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Prior work [Cruz-Filipe, Marques-Silva, Schneider-Kamp '19; Giljegård and Wennerbreck '21] verified specific encodings; our library is general

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Quick demo!

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Basis for future verification efforts

Goal: prove that an encoding is correct

Library preliminaries

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But what is a correct encoding?

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F encodes C if for all truth assignments τ,

$$C(\tau(x_1), \ldots, \tau(x_n)) \;\leftrightarrow\; \exists \sigma,\, \sigma(F) = \top,$$

where σ agrees with τ on X (i.e. $\forall x \in X,\, \tau(x) = \sigma(x)$)
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An encoding function E is correct for C if the formula it produces encodes C on all inputs

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In Lean, the definitions look like:

```
def encodes (C : constraint) (l : list literal) (F : cnf) :=
  ∀ (τ : assignment),
  (C.eval τ l = tt) ↔
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We prove that the encoding functions in our library are correct according to these definitions

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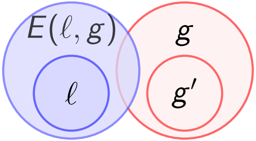

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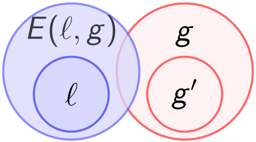

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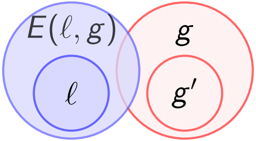

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  (enc l g).gensym ⊆ g ∧
  vars (enc l g).formula ⊆ (vars l) ∪ (g \ (enc l g).gensym)
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Case study: at-most-one

The Sinz at-most-one encoding produces $\sim 3n$ clauses and needs $n-1$ new variables:

$$\mathrm{Sinz}(X) = \bigwedge_{i=1}^{n-1} \left((\overline{x}_i \vee s_i) \wedge (\overline{s}_i \vee s_{i+1}) \wedge (\overline{s}_i \vee \overline{x}_{i+1}) \right)$$

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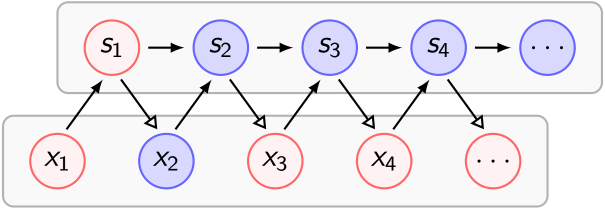

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In "direct" proofs, supply extended assignments explicitly

Applications and future work

Combine sub-encodings to form more complex ones

Easily recover proofs of correctness

```
def append (enc₁ enc₂ : enc_fn) : enc_fn :=
  λ (l : list literal) (g : gensym),
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Combine sub-encodings to form more complex ones

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Toy example by combining sub-encodings for Sudoku (demo!)

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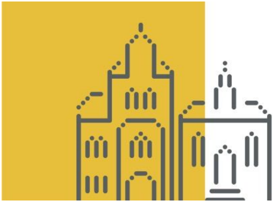
Overall, the goal is to make Lean the one-stop-shop for generating SAT queries in a trusted way

Verified encodings for SAT solvers



Thank you for your attention!
Any questions?