Verifying SAT Encodings in Lean

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Advised by Marijn J. H. Heule and Jeremy Avigad
Bryan Parno, committee member

Thesis at
http://crcodel.com/research/verified-encodings-thesis.pdf

Overview

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Our correctness proofs improve confidence in solver results.

We develop general proof methods and data structures.

What is an encoding?

The Lean theorem prover

Encoding verification library

Encodings and proofs of correctness

Conclusion

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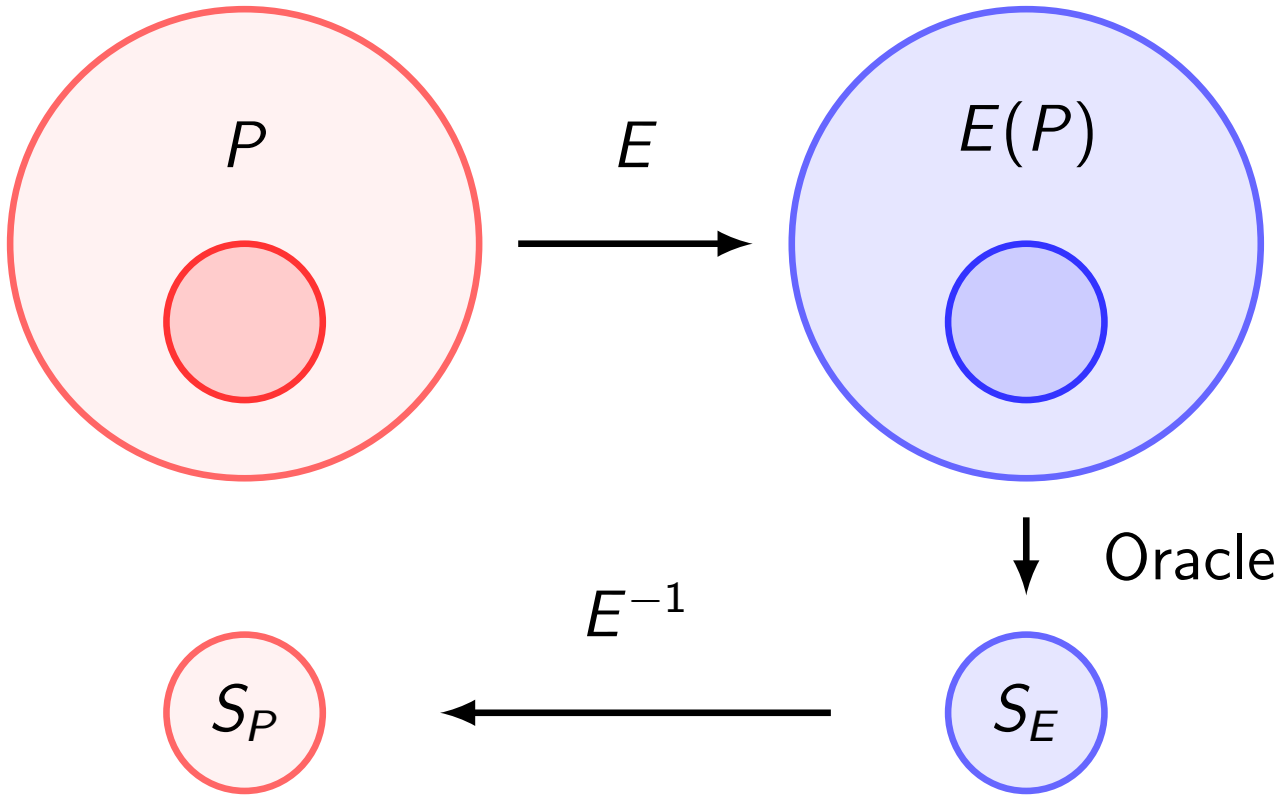

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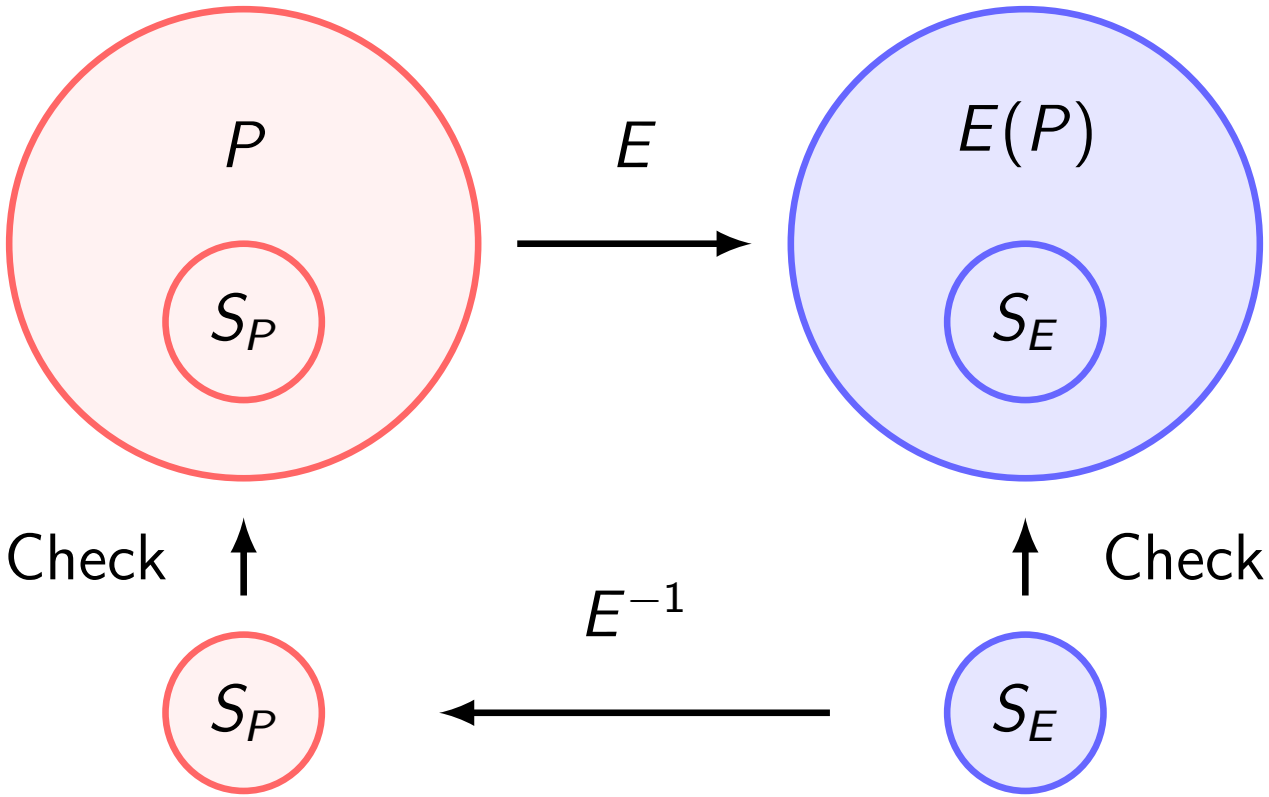

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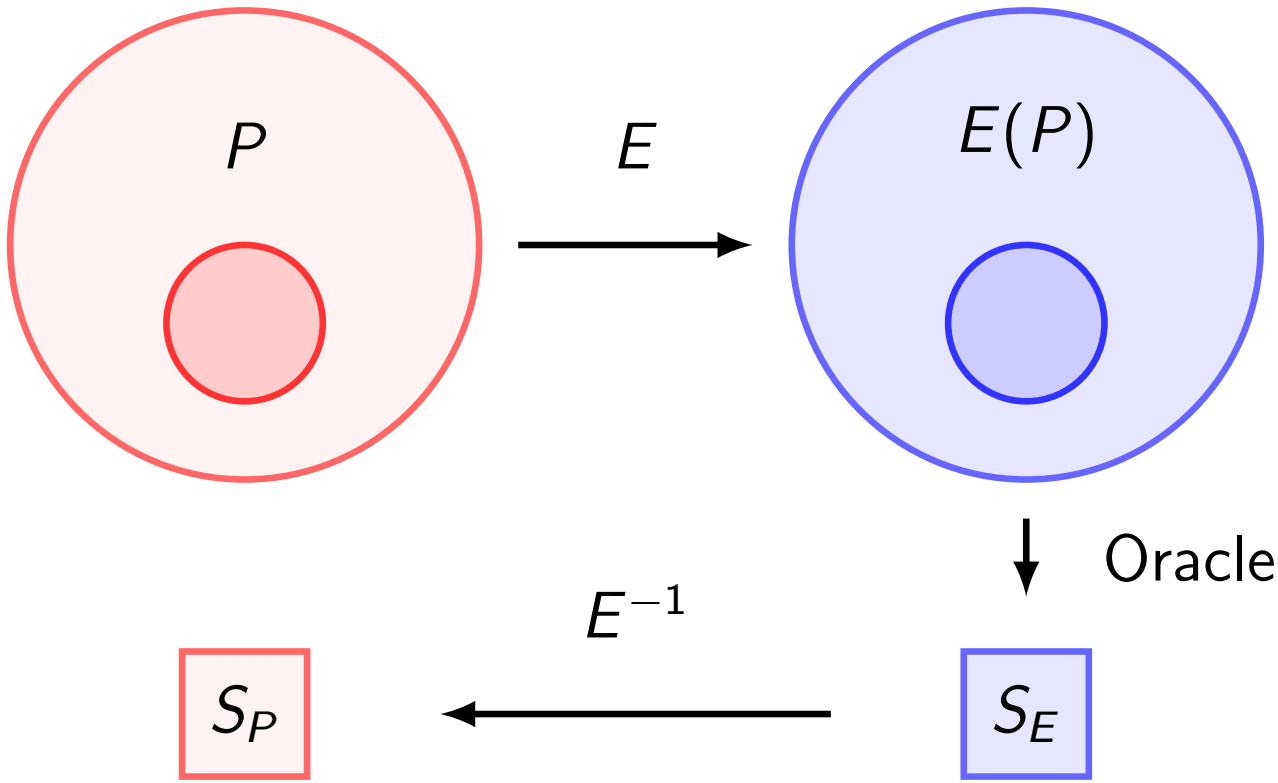

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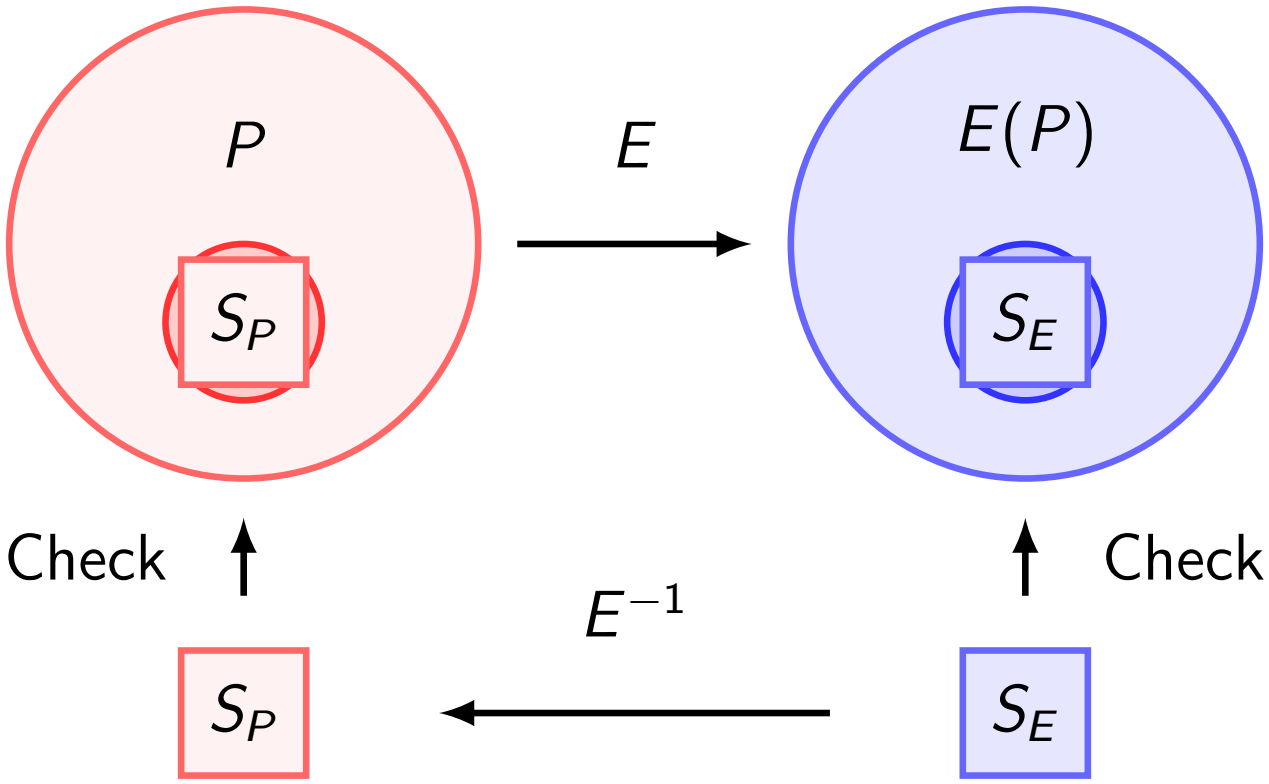

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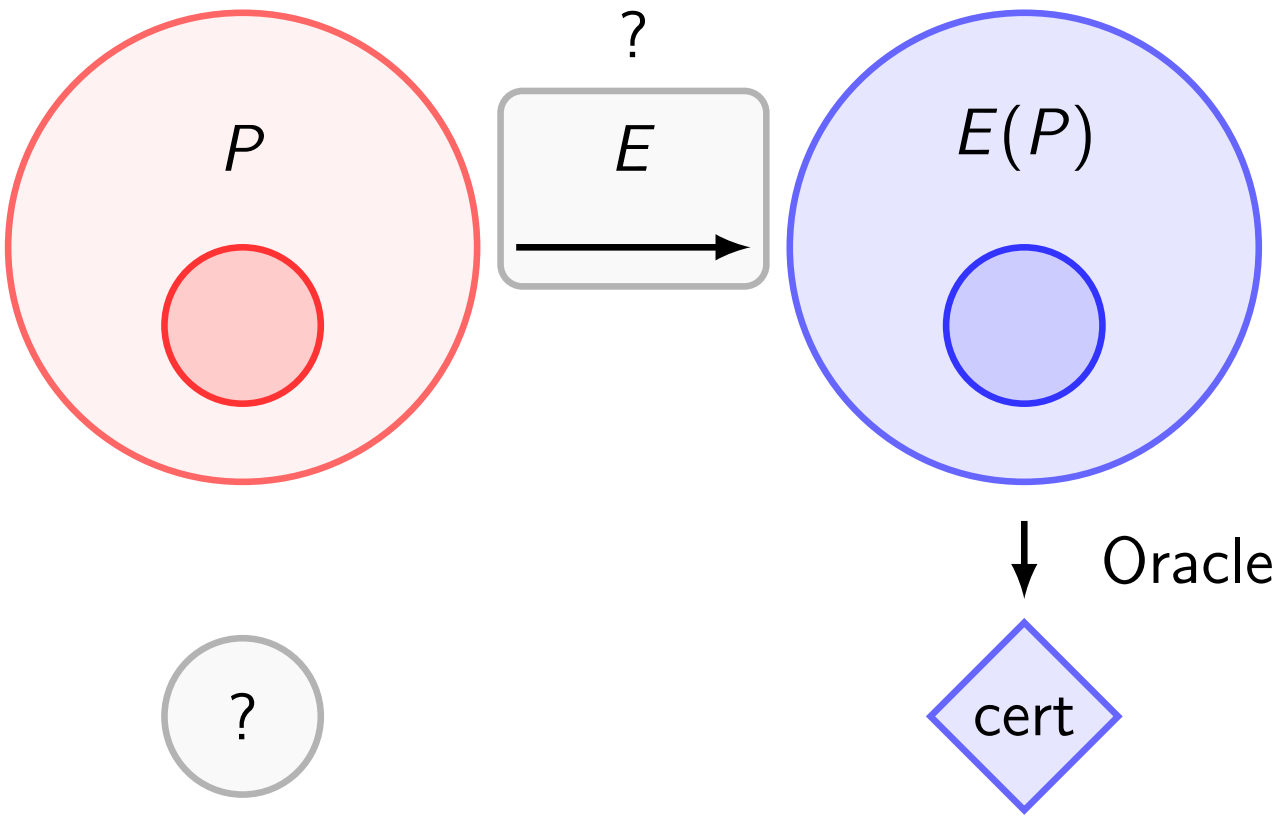

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τ satisfies F (written $\tau(F) = \top$) and F is satisfiable.

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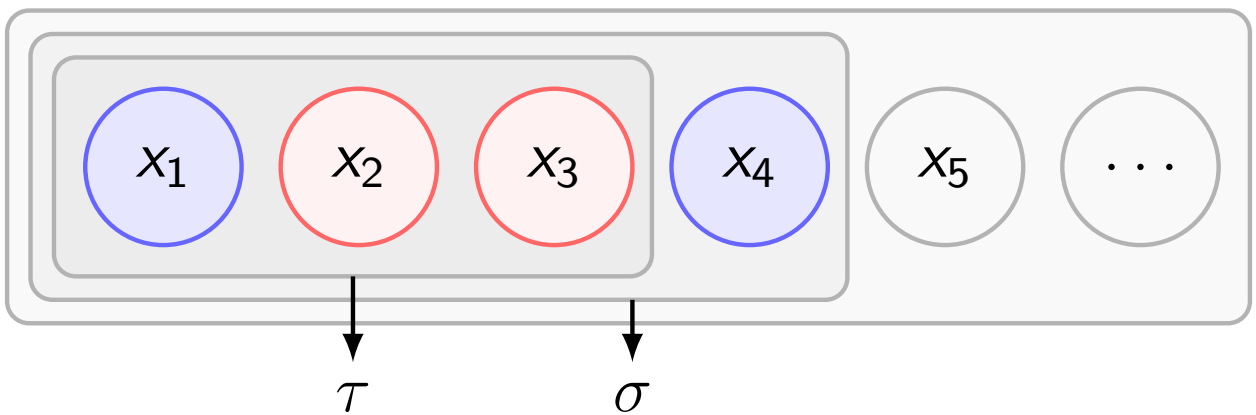
What is an encoding?

Definition: Extended truth assignment

Let τ be an assignment on $V(\tau)$. Then σ extends τ if $V(\tau) \subseteq V(\sigma)$ and $\forall x \in V(\tau),\ \tau(x) = \sigma(x)$.



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Definition: Encoding of a Boolean relation

Let R be a Boolean relation, F be a formula, and $x_1, \ldots, x_n$ be variables representing the inputs of R. Then F encodes R iff: $\forall \tau$ on $x_1, \ldots, x_n$, $R(\tau(x_1), \ldots, \tau(x_n))$ iff F is satisfied by some assignment that extends τ.

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Special case $R := (f(\tau(x_1), \ldots, \tau(x_n)) = \top)$. Then R is defined by f.

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The Lean theorem prover

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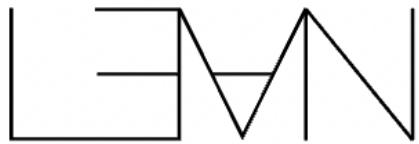

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We used Lean 3; Lean 4 is currently in development.

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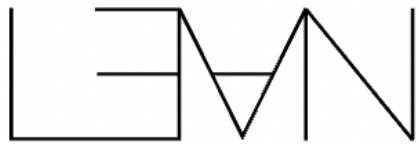

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Quick demo!

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Contains:

- Data structures (CNF representations, `gensym`)
- Supporting lemmas and theorems
- Encoding proofs of correctness

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Basis for future verification efforts

Encoding verification library - CNF representations

```
inductive literal (V : Type*)
| Pos (v : V) : literal
| Neg (v : V) : literal

def clause (V : Type*) := list (literal V)
def cnf (V : Type*) := list (clause V)
def assignment (V : Type*) := V -> bool
```

Encoding verification library - CNF representations

```
def eval (τ : assignment V) (c : clause V) : bool :=
  c.foldr (λ l b, b || (l.eval τ)) ff
```

`l.eval` is syntactic sugar for `literal.eval l`

`tt` is true, `ff` is false

Encoding verification library - CNF representations

```
theorem eval_tt_iff_exists_literal_eval_tt
  {τ : assignment V} {c : clause V} :
  c.eval τ = tt ↔ ∃ (l : literal V), l ∈ c → l.eval τ = tt

theorem eval_ff_iff_forall_literal_eval_ff
  {τ : assignment V} {c : clause V} :
  c.eval τ = ff ↔ ∀ (l : literal V), l ∈ c → l.eval τ = ff
```

Encoding verification library - `gensym`

```
structure gensym (α : Type *) := (offset : nat)
  (f : nat -> α) (f_inj : injective f)
```

Inspired by de Bruijn indices, Hilbert's hotel, Lisp gensym

Encoding verification library - `gensym`

```
def stock : set α := { a | ∃ (n : nat), g.f (g.offset + n) = a }
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Encoding verification library - gensym

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def stock : set α := { a | ∃ (n : nat), g.f (g.offset + n) = a }

def fresh (g : gensym α) : (α × gensym α) :=
  ⟨g.f (g.offset), ⟨g.offset + 1, g.f, g.f_inj⟩⟩

def nth (g : gensym α) (n : nat) : α := g.f (g.offset + n)
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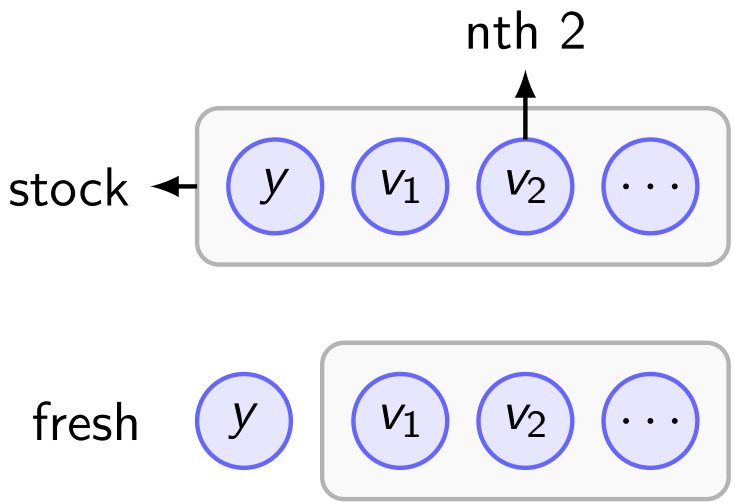

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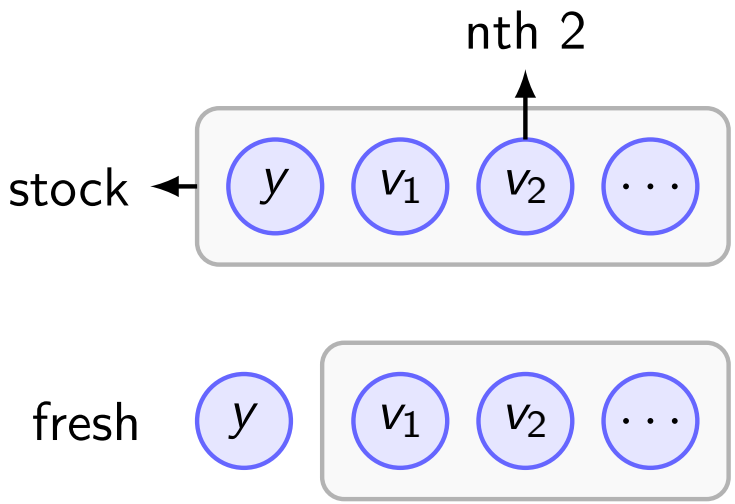

In practice, gensym is often a global int counter.

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Encodings and proofs of correctness

We proved correct:

- Two encodings of XOR (direct and Tseitin)
- Two encodings of AMO (direct and Sinz)
- One encoding of AMK (Sinz)

Encodings and proofs of correctness - XOR

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Important properties

$$\top \oplus x \equiv \overline{x} \qquad\qquad \bot \oplus x \equiv x$$

Encodings and proofs of correctness - XOR, DIRECTXOR

$$\textsc{directxor}(x_1, \ldots, x_n) = \bigwedge_{\text{even \# of negations}} \left(\pm x_1 \vee \cdots \vee \pm x_n \right)$$

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Generates 2^{n-1} clauses on n input variables.

Poor performance for solvers above small n.

Encodings and proofs of correctness - XOR, DIRECTXOR

```
def direct_xor : list (literal V) → cnf V
| []        := [[]]
| (l :: ls) := (explode (map var ls)).map
    (λ c, ite (!bodd (c.count_flips ls)) (l :: c) (l.flip :: c))
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$$\texttt{explode}([x, y, z]) = [[x, y, z], [\overline{x}, y, z], [x, \overline{y}, z], [x, y, \overline{z}], [\overline{x}, \overline{y}, \overline{z}], [x, \overline{y}, \overline{z}], [\overline{x}, y, \overline{z}], [\overline{x}, \overline{y}, z]]$$

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Add in `l` according to parity of clause.

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Theorem: DIRECTXOR encodes XOR.

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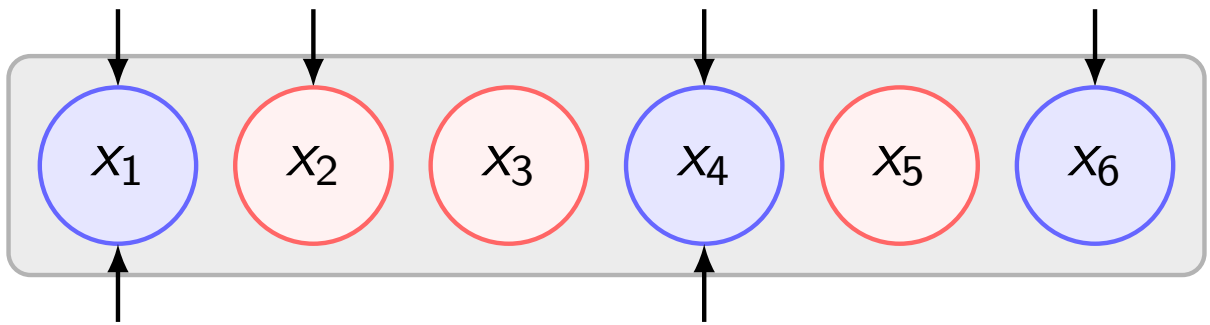

Odd number of the x_i true $\neq$ even number of negations. □

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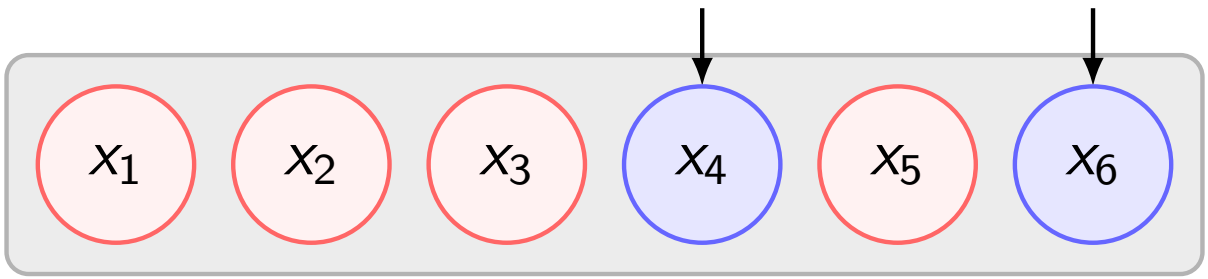

Identify the clause with exactly the true x_i negated. □

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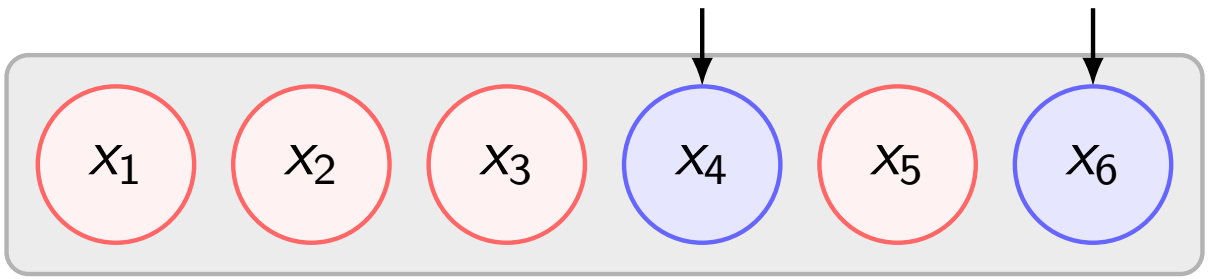

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Takeaway: clauses disallow falsifying inputs—stick to the semantics.

Encodings and proofs of correctness - XOR, Tseitin

Tseitin encoding (linear)

$$\text{LINEAR}_k(x_1, \ldots, x_n) =$$
$$\begin{cases} \text{DIRECTXOR}(x_1, \ldots, x_n) & n \leq k \\ \text{DIRECTXOR}(x_1, \ldots, x_{k-1}, \overline{y}) \wedge \text{LINEAR}_k(y, x_k, \ldots, x_n) & \text{o.w.} \end{cases}$$

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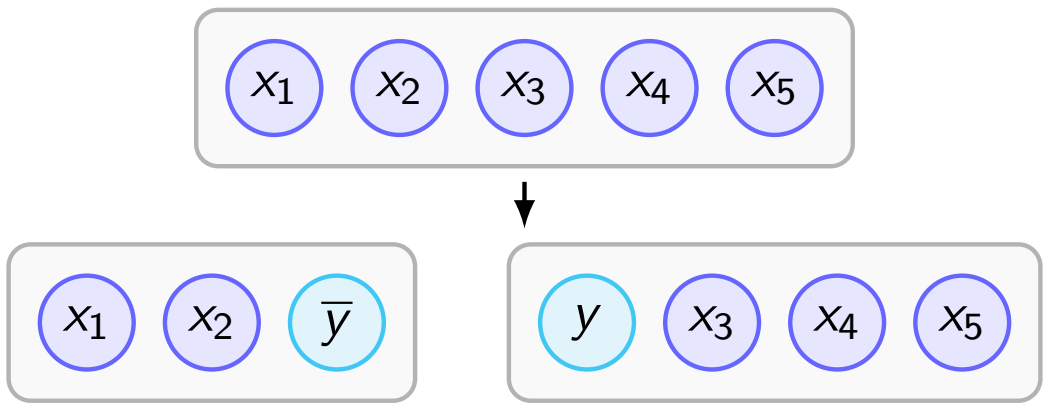

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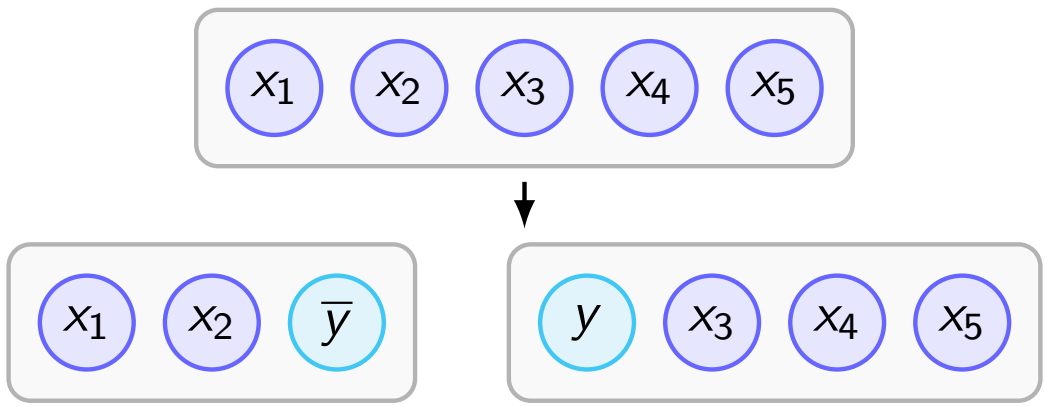

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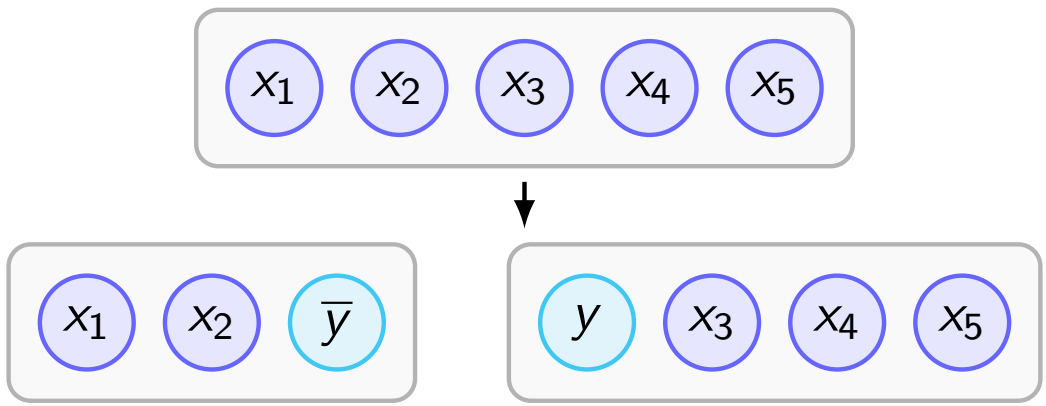

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Solver performance impacted by cutting number and linear vs. pooled.

Encodings and proofs of correctness - XOR, Tseitin

$$\text{LINEAR}_3(x_1, \ldots, x_6) =$$
$$D(x_1, x_2, \overline{y}_1) \wedge D(y_1, x_3, \overline{y}_2) \wedge D(y_2, x_4, \overline{y}_3) \wedge D(y_3, x_5, x_6)$$

$$\text{POOLED}_3(x_1, \ldots, x_6) =$$
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Linear requires $O(n)$ updates to the y_i, pooled only $O(\log n)$.

Encodings and proofs of correctness - XOR, Tseitin

```
def linear_xor {k : nat} (hk : k ≥ 3) :
  list (literal V) → gensym V → cnf V
| l g := ite (length l ≤ k) (direct_xor l)
    direct_xor (l.take (k - 1) ++ [Neg g.fresh.1]) ++
    linear_xor (Pos g.fresh.1 :: l.drop (k - 1)) g.fresh.2)
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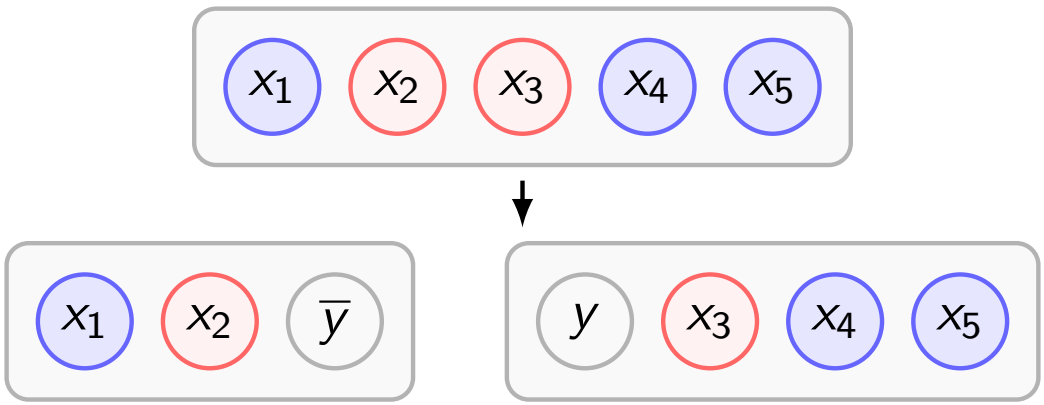
Challenge: generating fresh variables - use `fresh`.

Encodings and proofs of correctness - XOR, Tseitin

Theorem: The Tseitin transformation encodes XOR.

Proof.

By strong induction on n.



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Encodings and proofs of correctness - XOR, Tseitin

Theorem: The Tseitin transformation encodes XOR.

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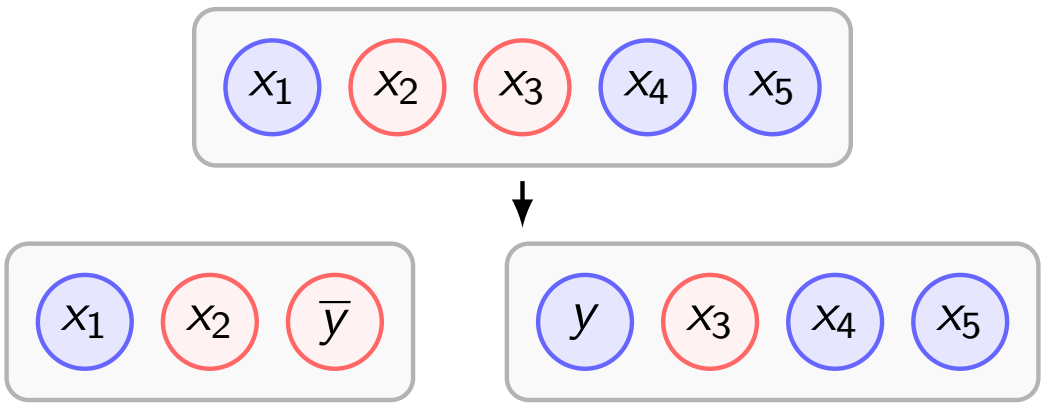

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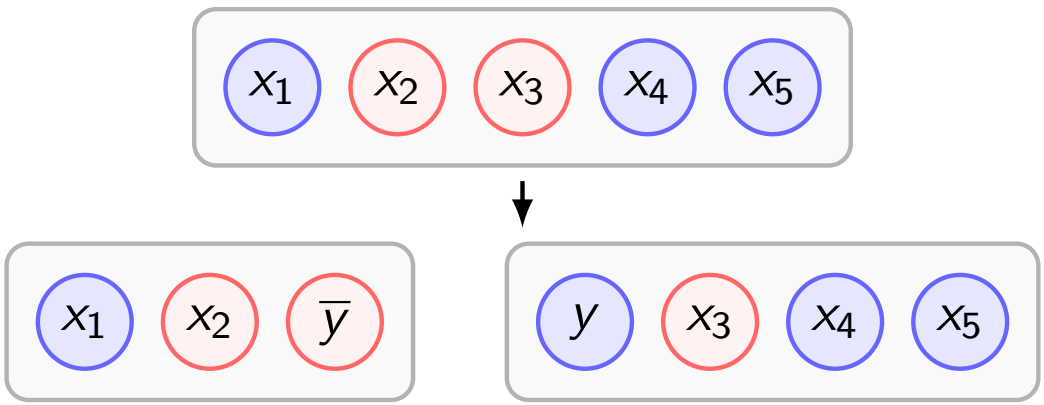

□

Takeaway: recursive definitions require inductive proofs.

`fresh` used to generate variables, updated `gensym` passed to recursive call.

Encodings and proofs of correctness - AMO

Definition

$$\text{AMO}(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i \leq 1$$

Encodings and proofs of correctness - AMO

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Part of general class of cardinality constraints.

$$\text{AMK}_k(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i \leq k.$$

Encodings and proofs of correctness - AMO, DIRECTAMO

$$\text{DIRECTAMO}(x_1, \ldots, x_n) = \bigwedge_{1 \leq i < j \leq n} (\overline{x}_i \vee \overline{x}_j)$$

Encodings and proofs of correctness - AMO, DIRECTAMO

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Generates $\binom{n}{2} \in O(n^2)$ clauses.

Encodings and proofs of correctness - AMO, DIRECTAMO

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def direct_amo : list (literal V) → cnf V
| []       := []
| (l::ls) := (ls.map (λ m, [l.flip, m.flip])) ++ (direct_amo ls)
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Recursive in nature, but global definition possible too

Correctness proof analogous to DIRECTXOR.

Developed `distinct` object to relate to semantics - crucial for proof.

Encodings and proofs of correctness - AMO, SINZAMO

$$\text{SINZAMO}(x_1, \ldots, x_n) =$$
$$\left(\bigwedge_{i=1}^{n-1}(\overline{x}_i \vee s_i)\right) \wedge \left(\bigwedge_{i=1}^{n-1}(\overline{s}_i \vee \overline{x}_{i+1})\right) \wedge \left(\bigwedge_{i=1}^{n-2}(\overline{s}_i \vee s_{i+1})\right)$$

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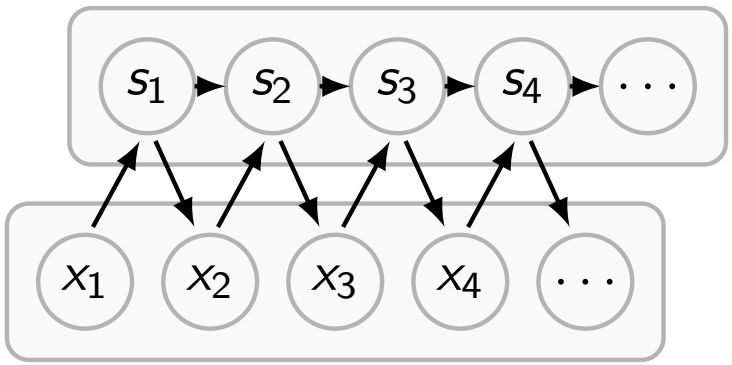

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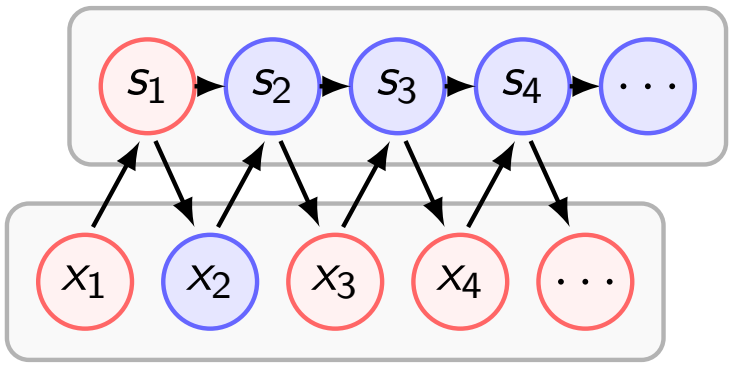

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Non-recursive definition - better for capturing global behavior.

Use `nth` instead of `fresh` to generate variables.

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Theorem: SINZAMO encodes AMO

Proof.

Provide explicit truth assignment. Give semantic meaning to each fresh variable.

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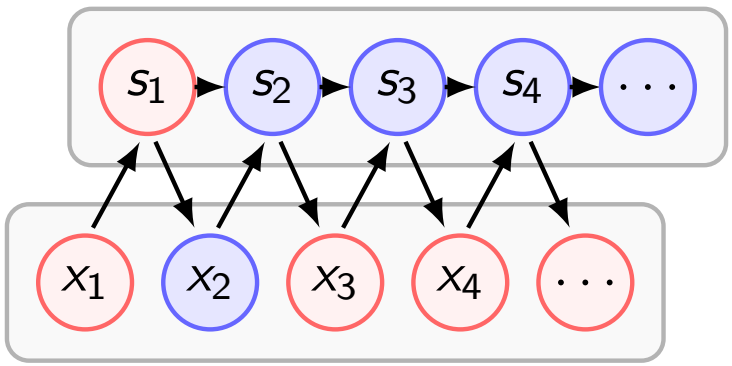

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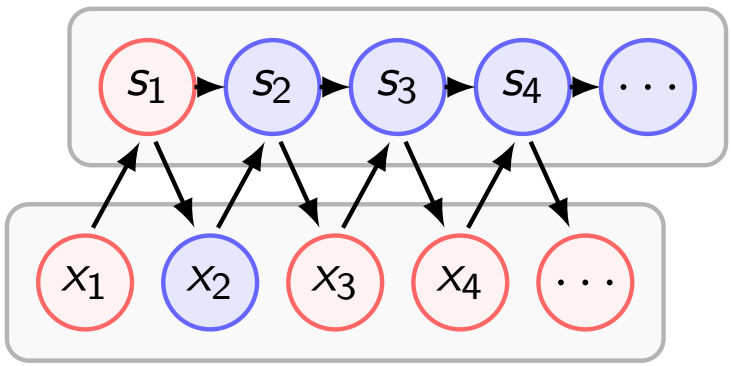

For reverse direction, take σ and show this explicit assignment works. $\square$

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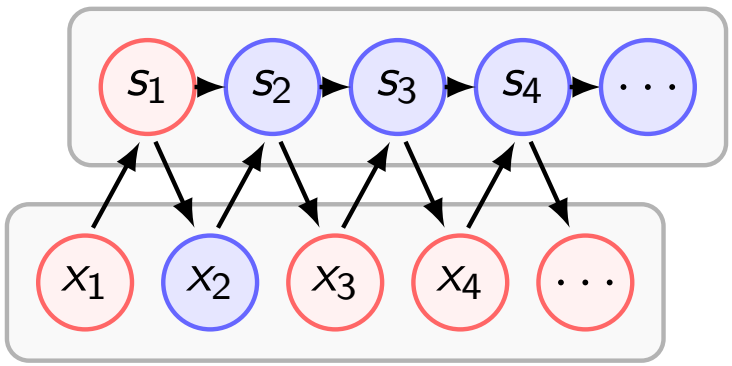

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Takeaway: For encoding with global behavior, tie the introduced variables to a semantic definition, and hard-code in a satisfying assignment

Encodings and proofs of correctness - AMK, Sinz

$$\textsc{SinzAMK}(x_1, \ldots, x_n) = \left(\bigwedge_{i=1}^{n}(\overline{x}_i \vee s_{i,1})\right) \wedge \left(\bigwedge_{i=1}^{n-1}\bigwedge_{j=1}^{k+1}(\overline{s}_{i,j} \vee s_{i+1,j})\right) \wedge \left(\bigwedge_{i=1}^{n-1}\bigwedge_{j=1}^{k}(\overline{x}_{i+1} \vee \overline{s}_{i,j} \vee s_{i+1,j+1})\right) \wedge \overline{s}_{n,k+1}$$

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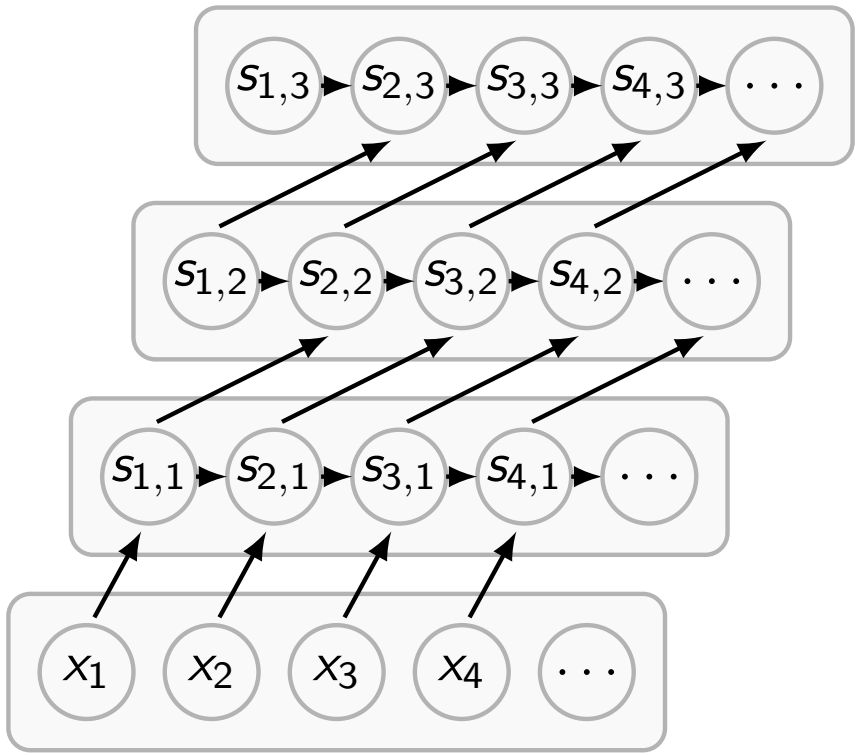

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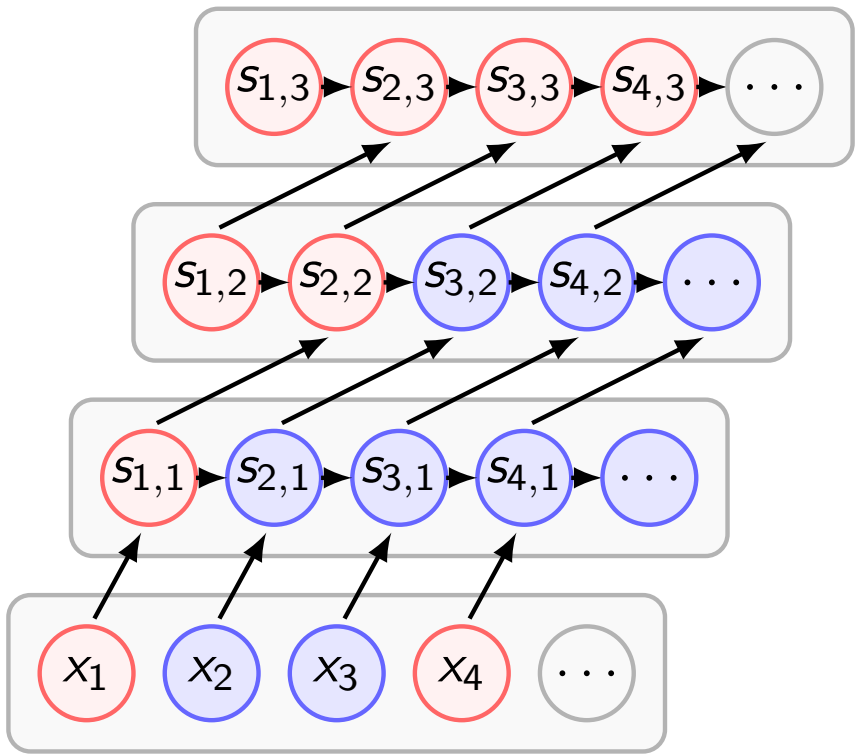

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def sinz_amk (k : nat) := ite (l = []) []
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Takeaway: Again, tie fresh variables to semantic meaning.

Conclusion

We prove the correctness of several common SAT encodings.

Fresh variable generation and management were challenging—`gensym` is our solution to the problems posed.

The proof methods (recursive vs. non-recursive) are general and will aid future verification efforts.

Future work: verify more encodings.