Formal Verification of the Empty Hexagon Number

Bernardo Subercaseaux[1], Wojciech Nawrocki[1], James Gallicchio[1], Cayden Codel[1], Mario Carneiro[1], Marijn J. H. Heule[1]

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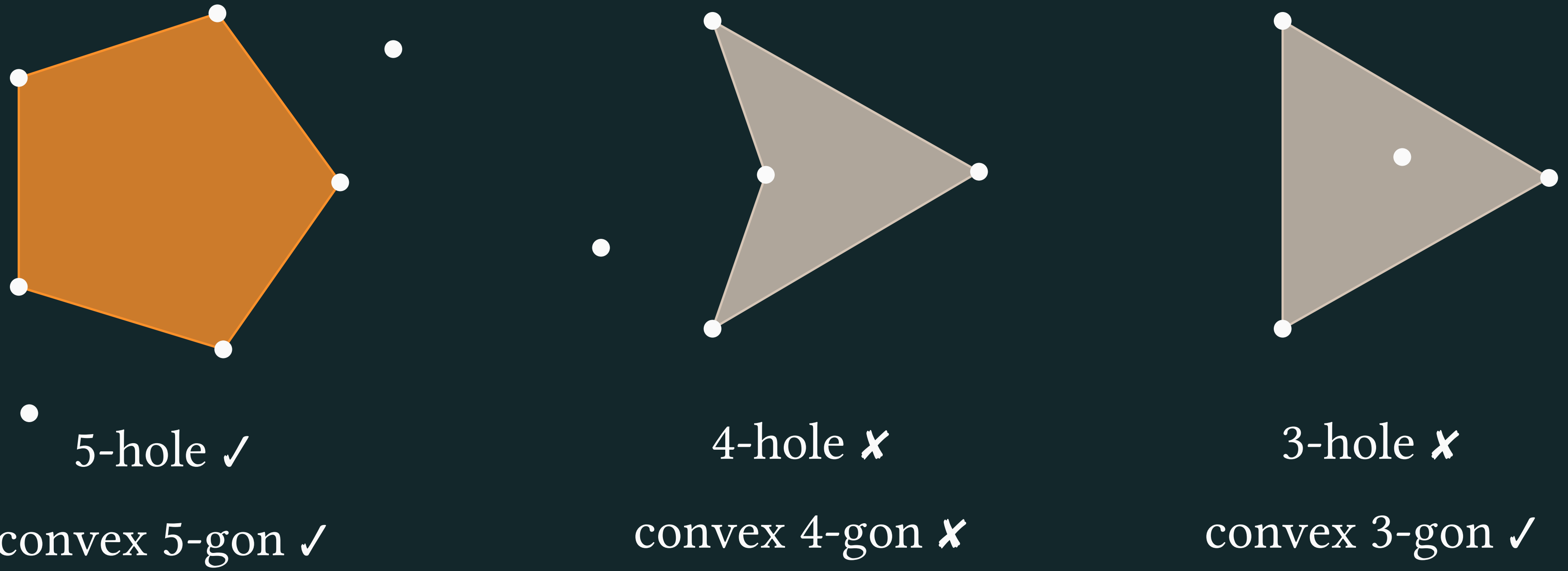
Tbilisi, Georgia

[1] Carnegie Mellon University, USA

Empty k-gons

Fix a set S of points on the plane, *no three collinear*. A ***k*-hole** is a convex k-gon with no point of S in its interior.



5 points must contain a 4-hole

Theorem (Klein 1932). Every set of 5 points in the plane contains a 4-hole.

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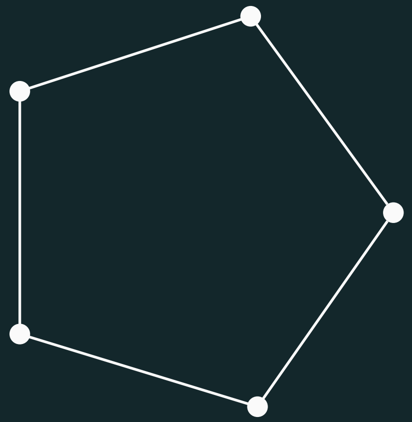

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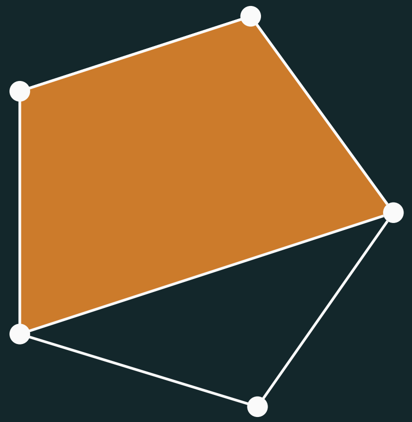

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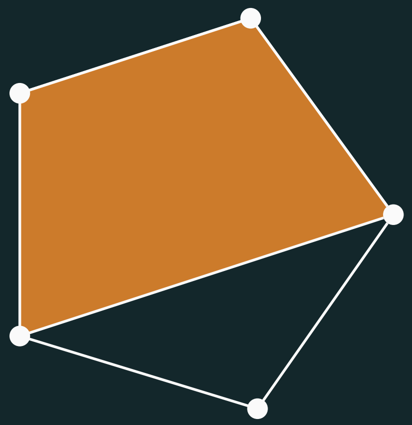

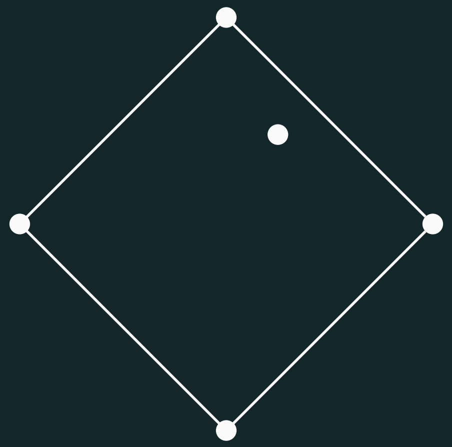

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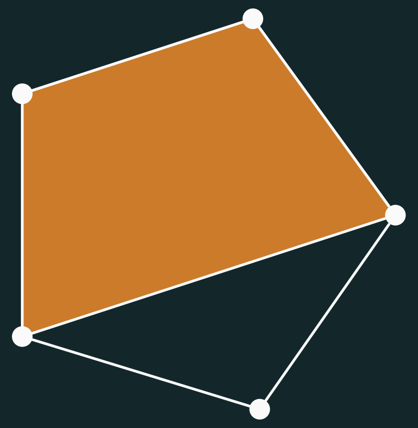

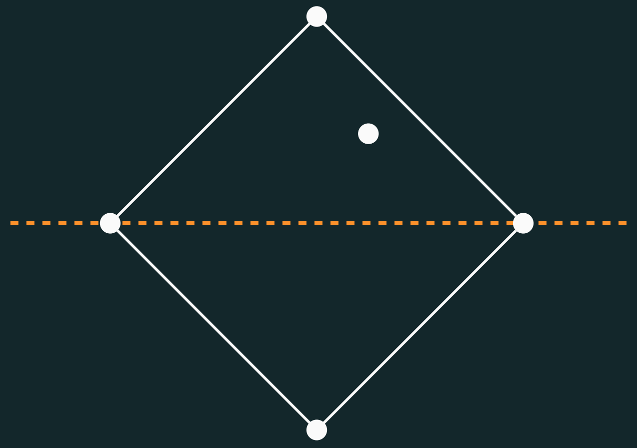

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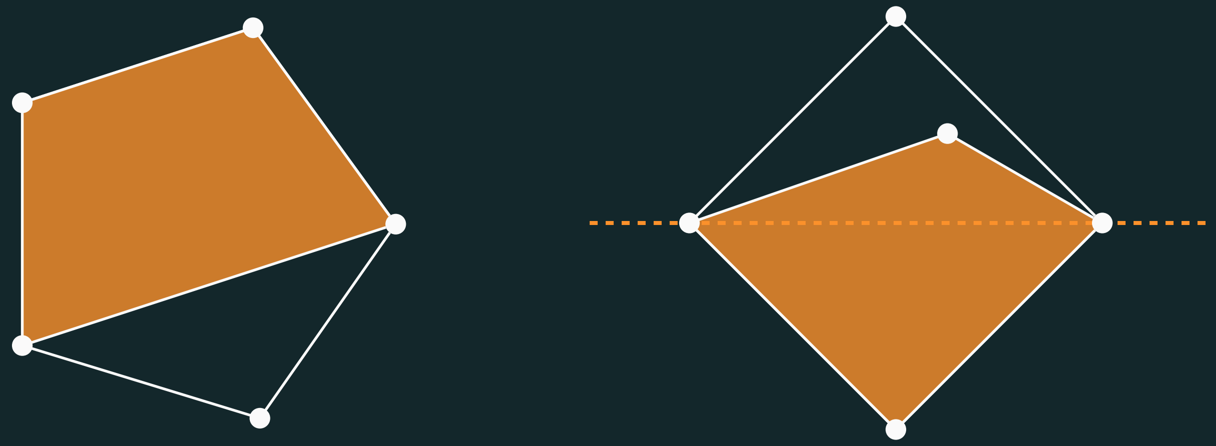

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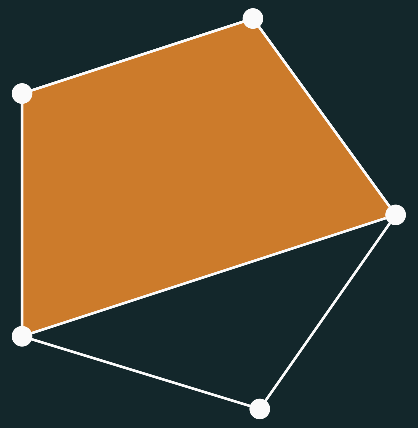

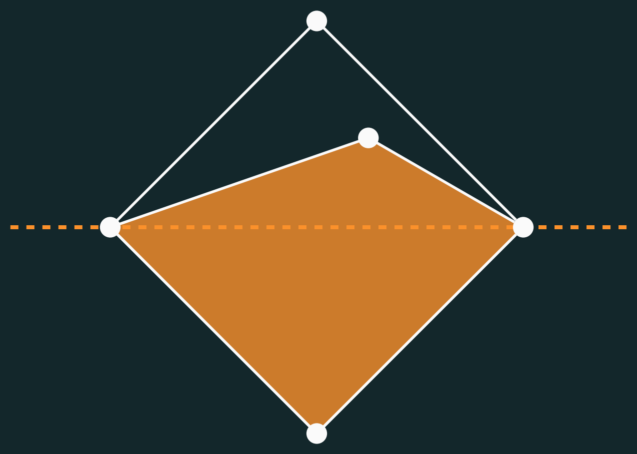

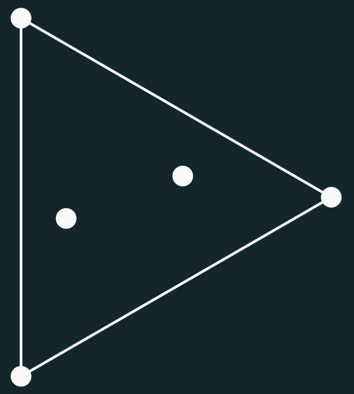

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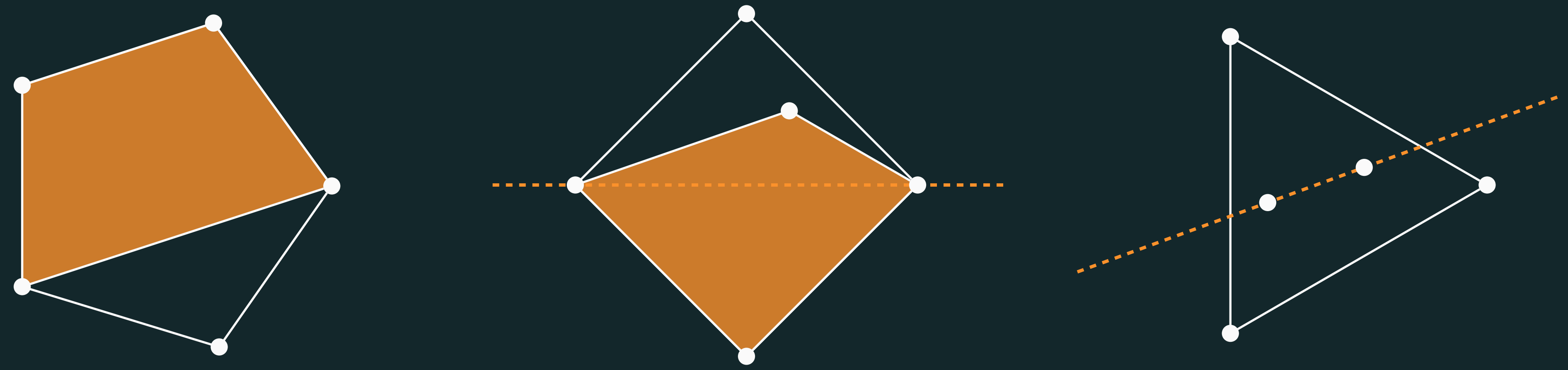

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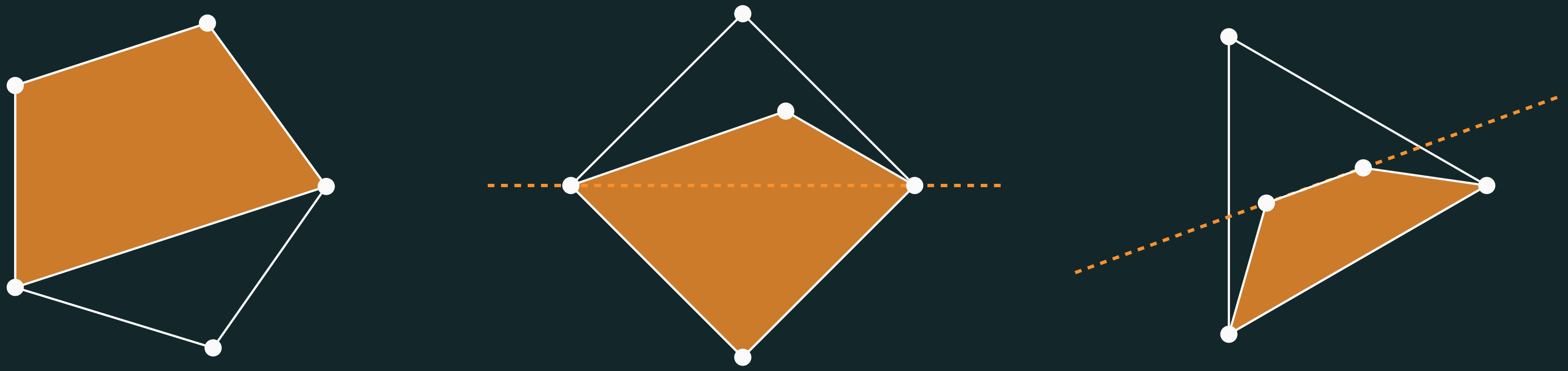

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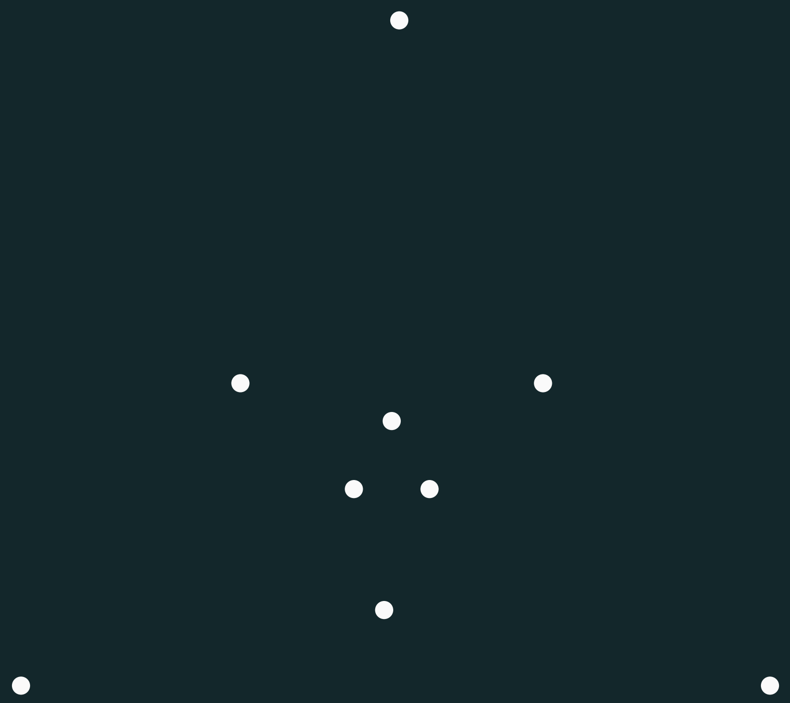

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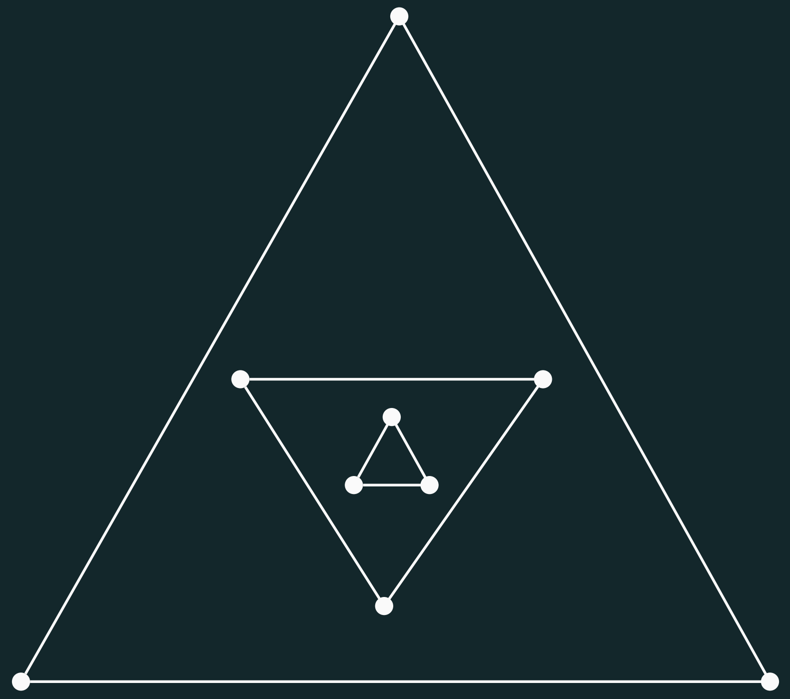

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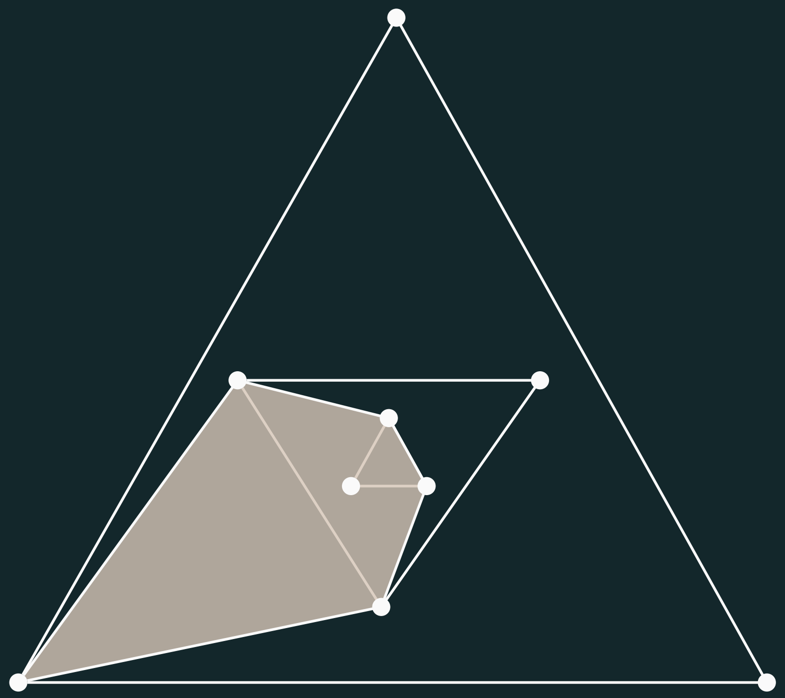

5-hole ✘

convex 5-gon ✓

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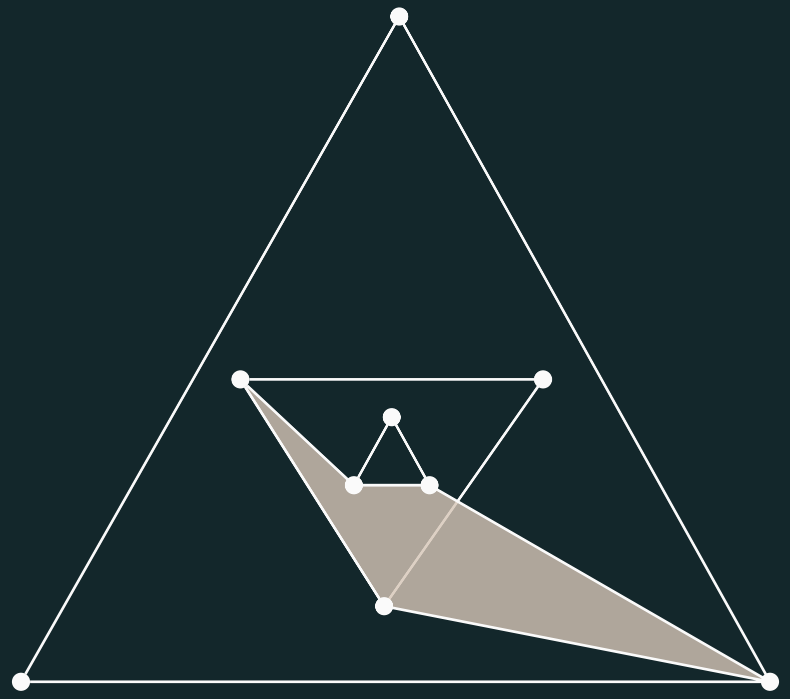

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The Happy Ending Problem

$g(k)$ = least n s.t. any set of n points must contain a **convex *k*-gon**

$h(k)$ = least n s.t. any set of n points must contain a ***k*-hole**

We just showed $h(4) \leq 5$ and $9 < h(5)$

Theorem (Erdős and Szekeres 1935). For a fixed k, every *sufficiently large* set of points contains a convex k-gon. So all $g(k)$ are finite.

Theorem (Horton 1983). For any $k \geq 7$, there exist arbitrarily large sets of points containing no k-holes. So $h(7) = h(8) = \ldots = \infty$.

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Known tight bounds

$h(3) = 3$ (trivial)
$h(4) = 5$ (Klein 1932)
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$h(6) = 30$ (Overmars 2003; Heule and Scheucher 2024)

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We formally verified all the above in Lean.

Upper bounds by combinatorial reduction to SAT.

- We focused on $h(6)$, established this year.
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Lower bounds by checking concrete sets of points.

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$2 < h(3)$ (trivial)
$4 < h(4)$ (Klein 1932)
$9 < h(5)$ (Harborth 1978)
$29 < h(6)$ **(Overmars 2003; Heule and Scheucher 2024)**

$2 < g(3)$ (trivial)
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Reduction. Show that a mathematical theorem is true if a propositional formula φ is unsatisfiable.

Solving. Show that φ is indeed unsatisfiable using a SAT solver.

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Reduction from geometry to SAT

1. Discretize from continuous coordinates in $\mathbb{R}^2$ to boolean variables.

2. Points can be put in *canonical form* without removing k-holes.

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theorem symmetry_breaking {l : List Point} :
  3 ≤ l.length → PointsInGenPos l →
  ∃ w : CanonicalPoints, l ≼σ w.points
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3. n points in canonical form with no 6-holes induce a propositional assignment that satisfies φ_n.

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theorem satisfies_hexagonEncoding {w : CanonicalPoints} :
  ¬σHasEmptyKGon 6 w → w.toPropAssn ⊨ Geo.hexagonCNF w.len
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4. But φ_{30} is unsatisfiable.

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axiom unsat_6hole_cnf : (Geo.hexagonCNF 30).isUnsat
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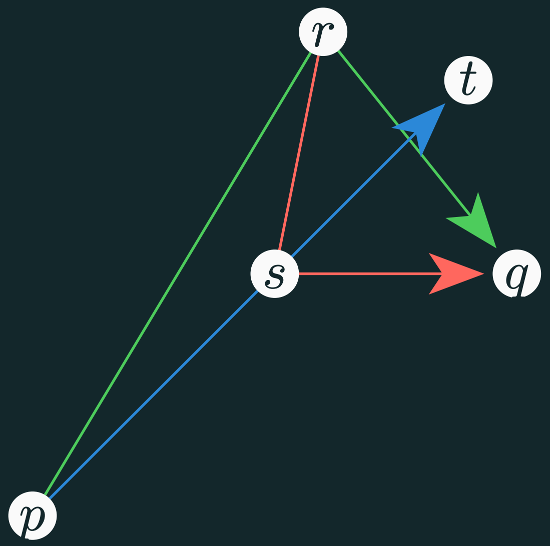
Discretization with triple-orientations



$$\sigma(p, r, q) = 1 \quad \text{(clockwise)}$$
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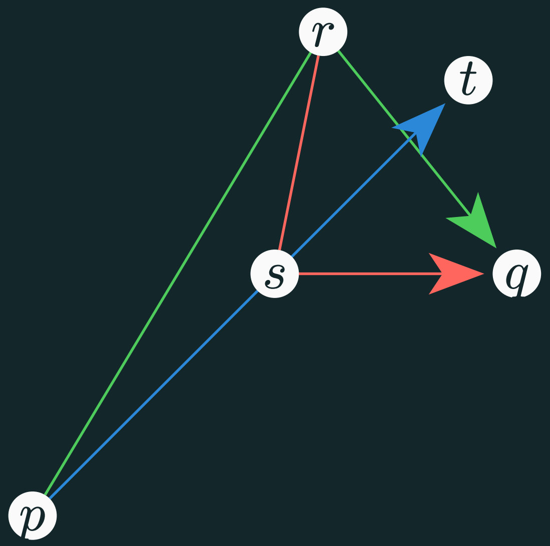

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$\exists$ k-hole $\iff$ a propositional formula over $\sigma(a, b, c)$ is satisfiable

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Symmetry breaking

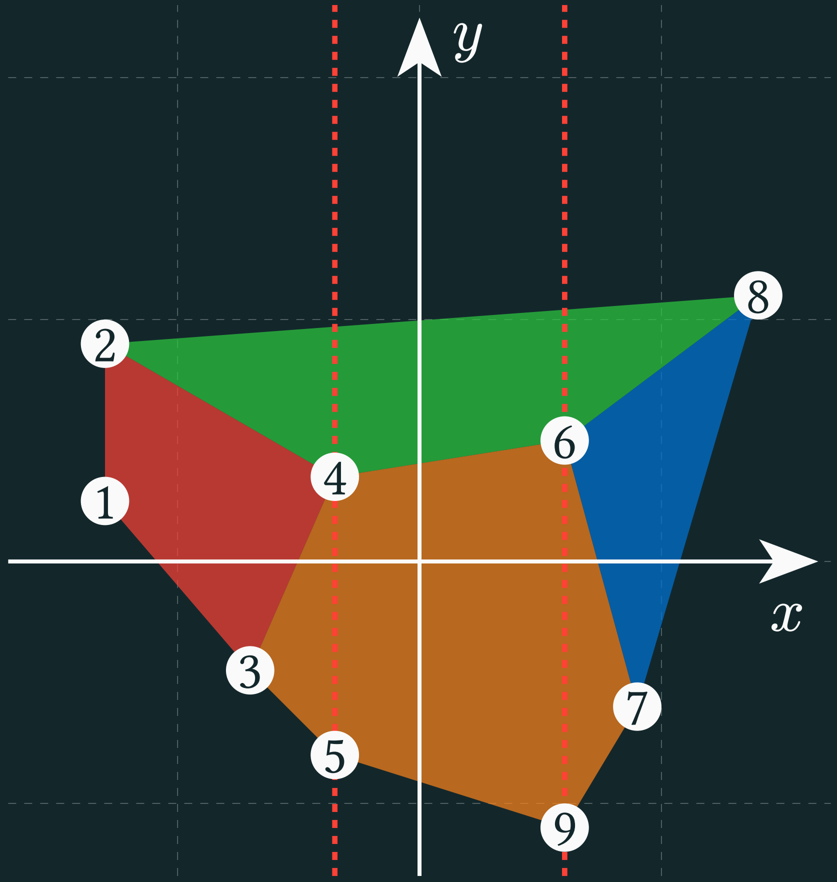
Lemma. WLOG we can assume that the points $(p_1, \ldots, p_n)$ are in *canonical form*, meaning that they satisfy the following properties:

- **(x-order)** The points are sorted with respect to their x-coordinates, i.e., $(p_i)_x < (p_j)_x$ for all $1 \leq i < j \leq n$.
- **(CCW-order)** All orientations $\sigma(p_1, p_i, p_j)$, with $1 < i < j \leq n$, are counterclockwise.
- **(Lex order)** The first half of list of adj. orientations is lex-below the second half:

$$\left[\sigma\left(p_{\lceil\frac{n}{2}\rceil+1}, p_{\lceil\frac{n}{2}\rceil+2}, p_{\lceil\frac{n}{2}\rceil+3}\right), \ldots, \sigma(p_{n-2}, p_{n-1}, p_n)\right] \succcurlyeq \left[\sigma\left(p_{\lceil\frac{n}{2}\rceil-1}, p_{\lceil\frac{n}{2}\rceil}, p_{\lceil\frac{n}{2}\rceil+1}\right), \ldots, \sigma(p_2, p_3, p_4)\right]$$

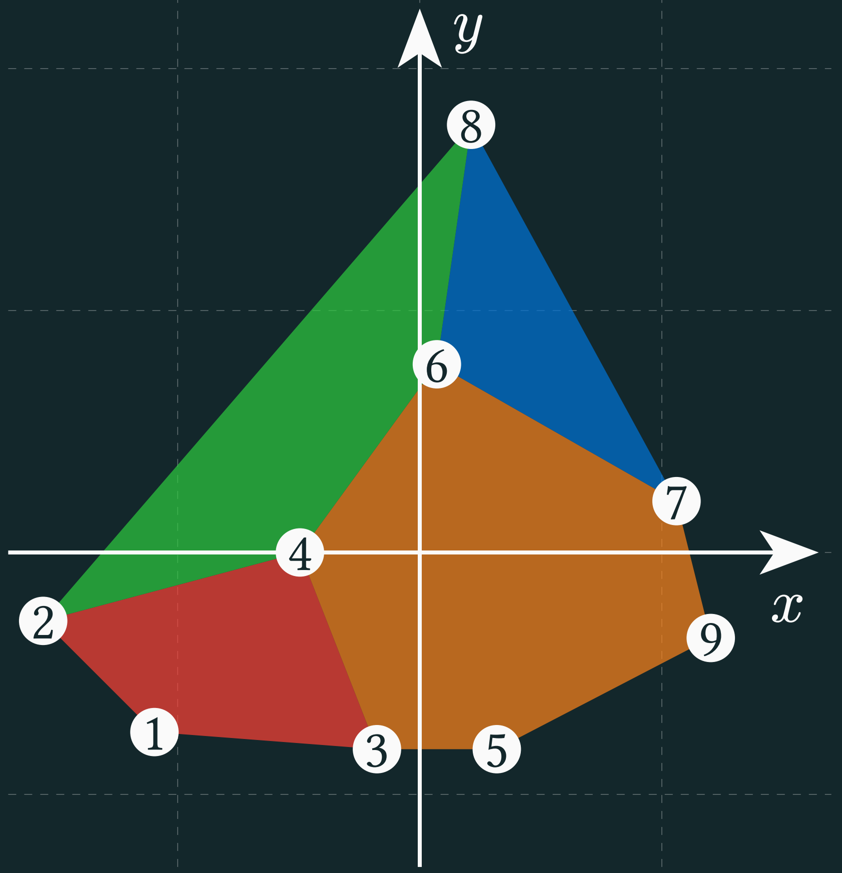
Symmetry breaking

Starting set of points.



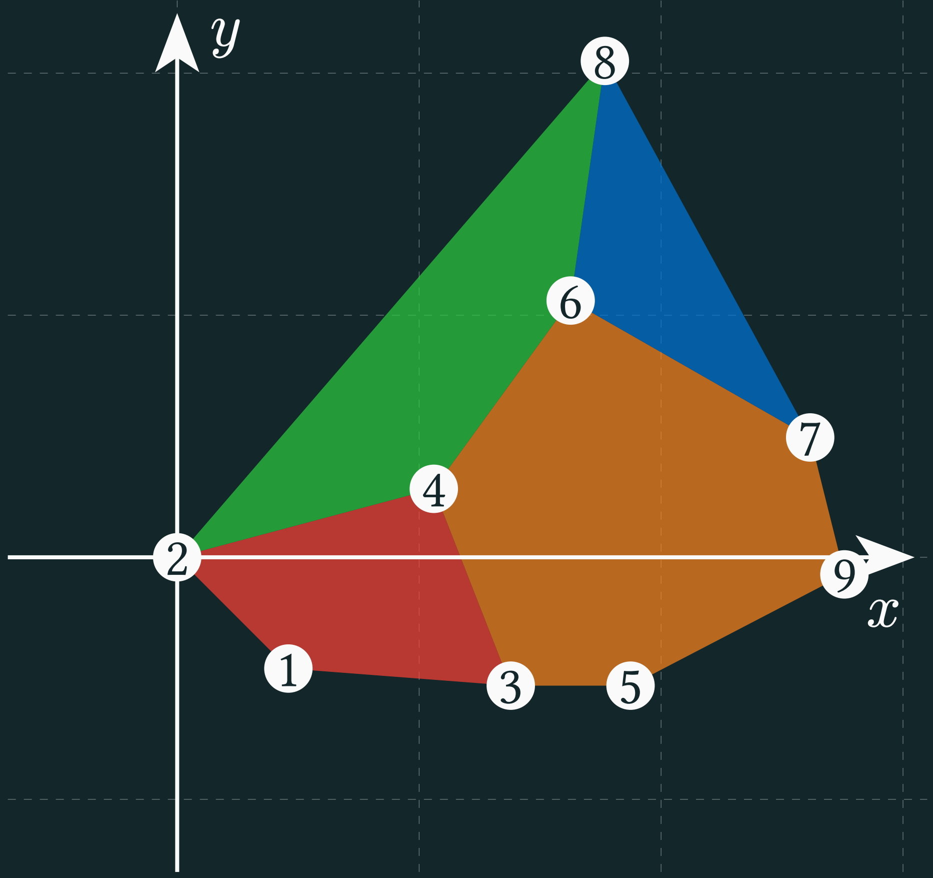
Symmetry breaking

Rotation ensures distinct x.



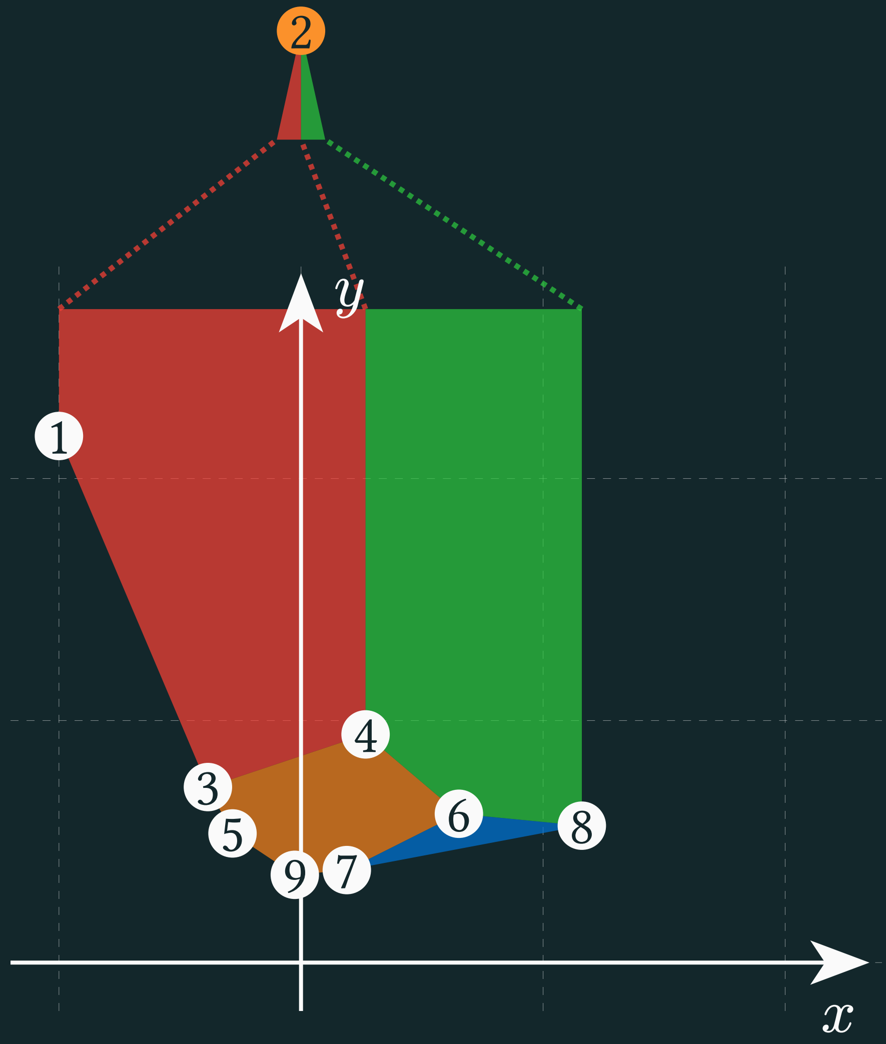
Symmetry breaking

Translate leftmost point to $(0, 0)$.
Ensures nonnegative x.



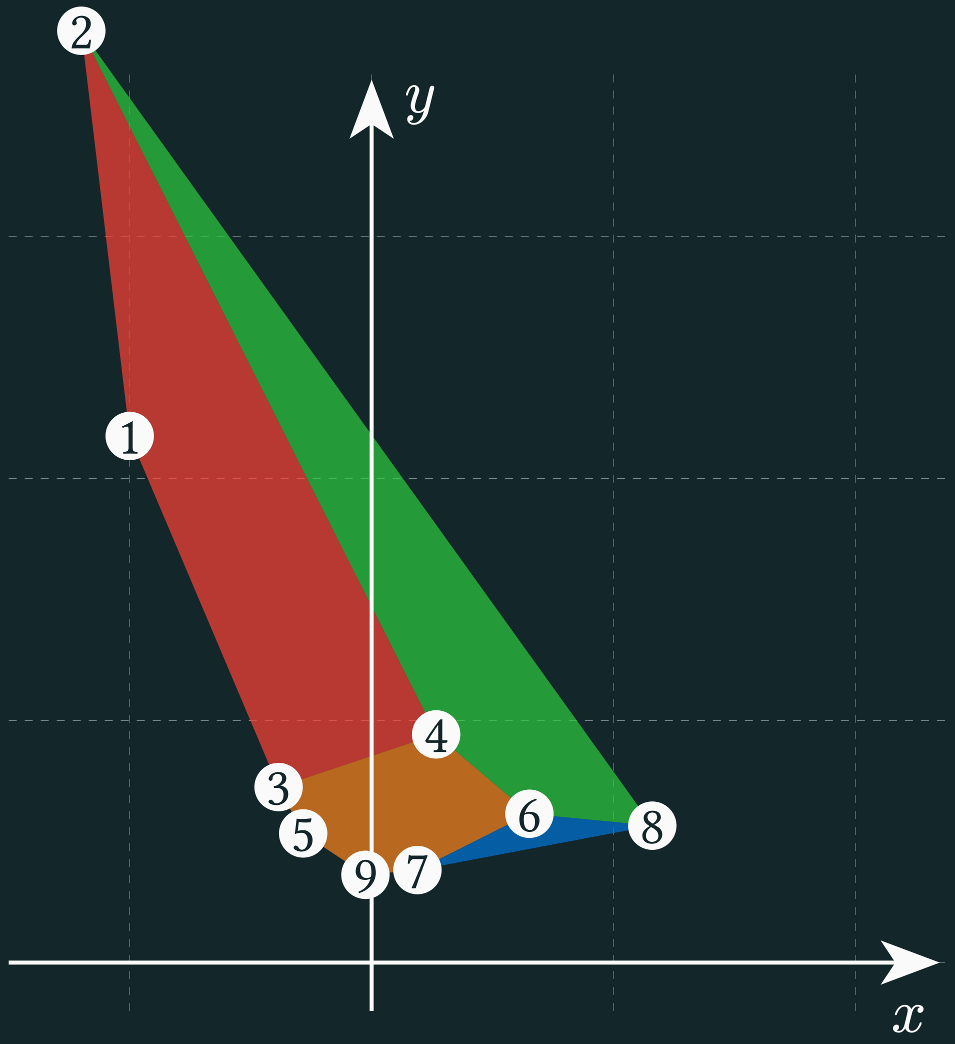
Symmetry breaking

Map $(x, y) \mapsto \left(\frac{y}{x}, \frac{1}{x}\right)$.



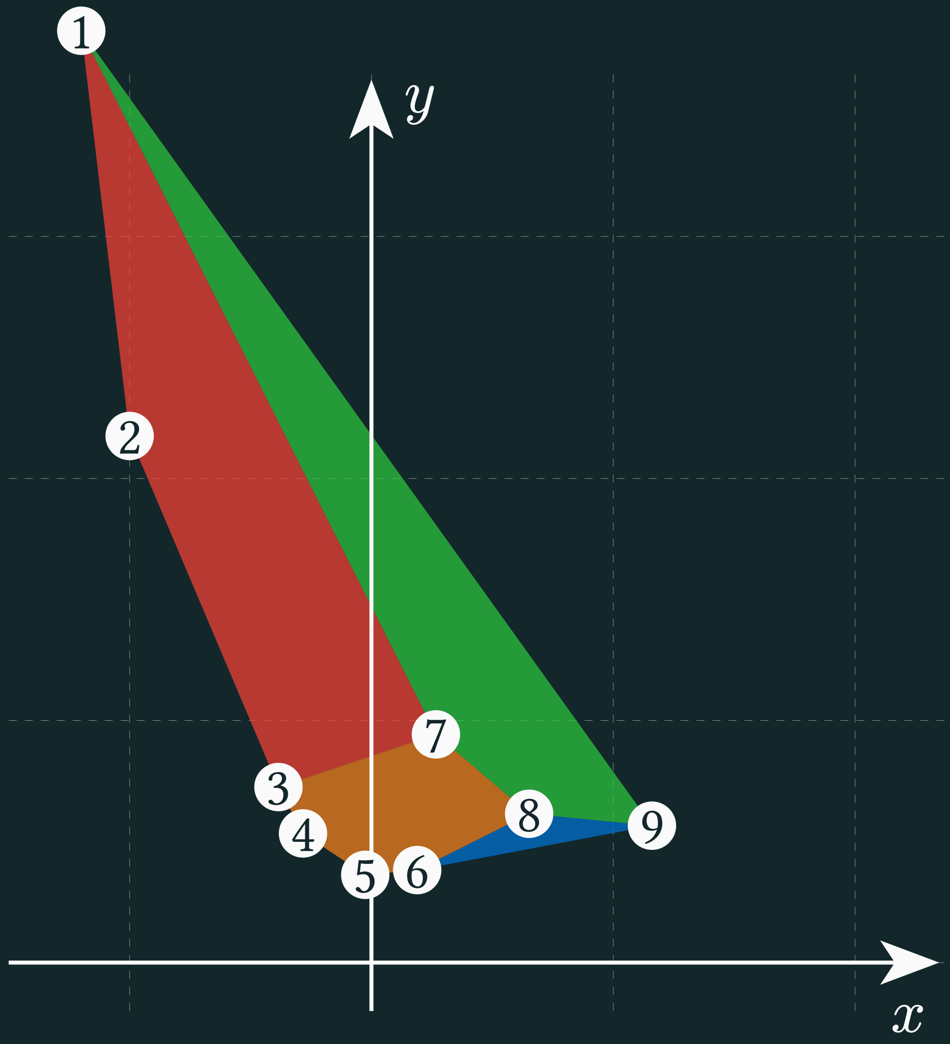
Symmetry breaking

Bring point at ∞ back.



Symmetry breaking

Relabel in order of increasing x.



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We verified an $\mathcal{O}\left(n^3\right)$ solution
from Dobkin, Edelsbrunner, and Overmars (1990).

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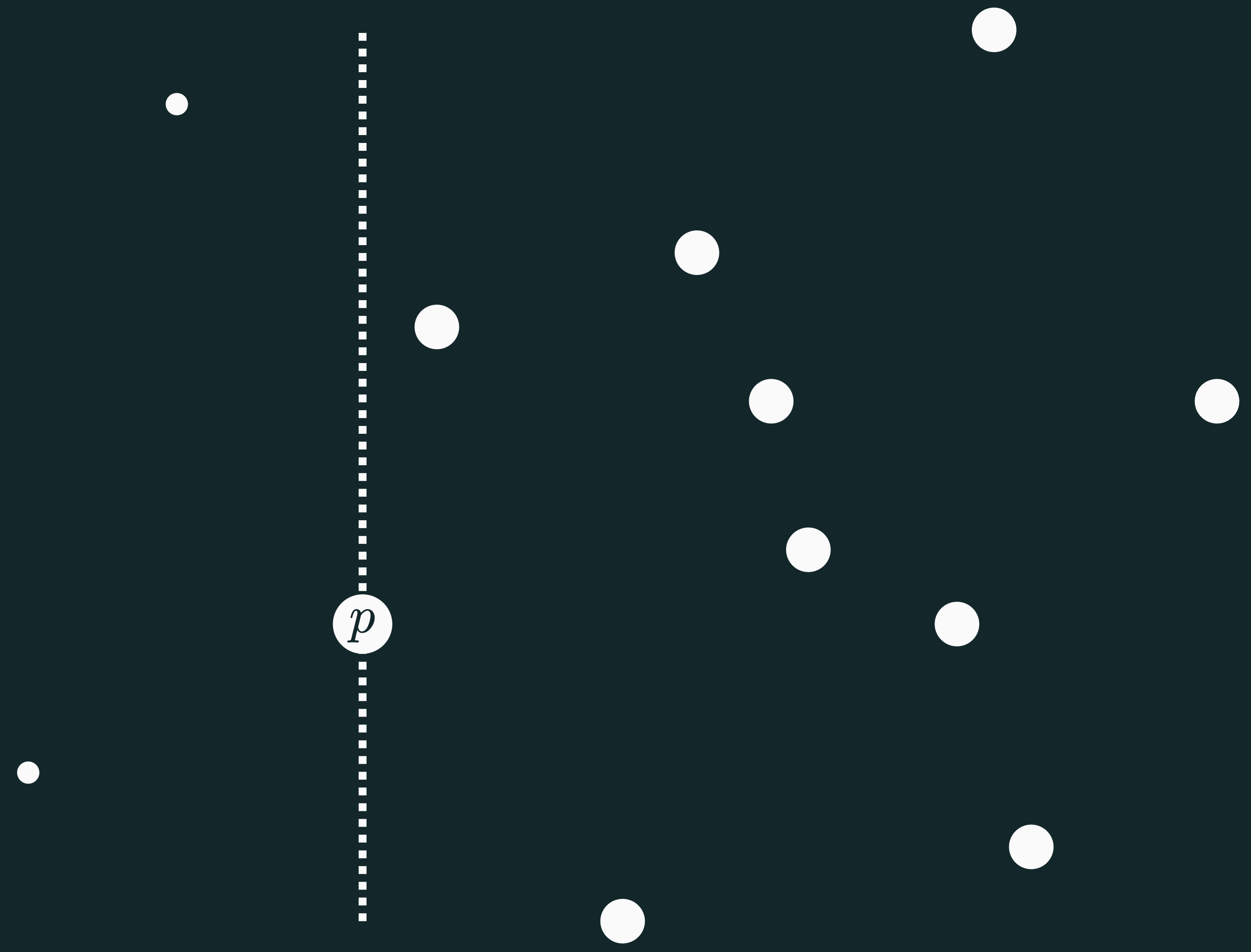

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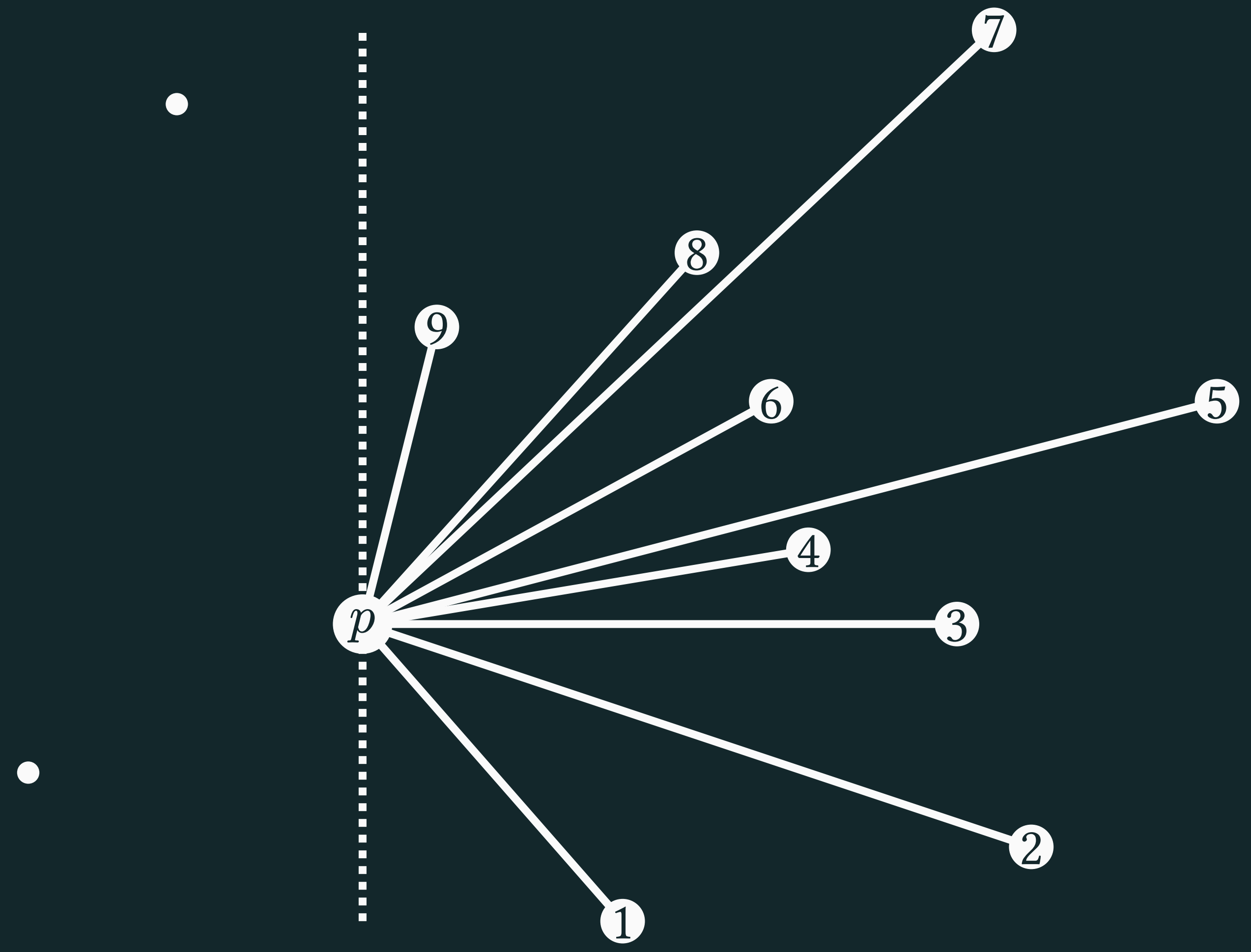

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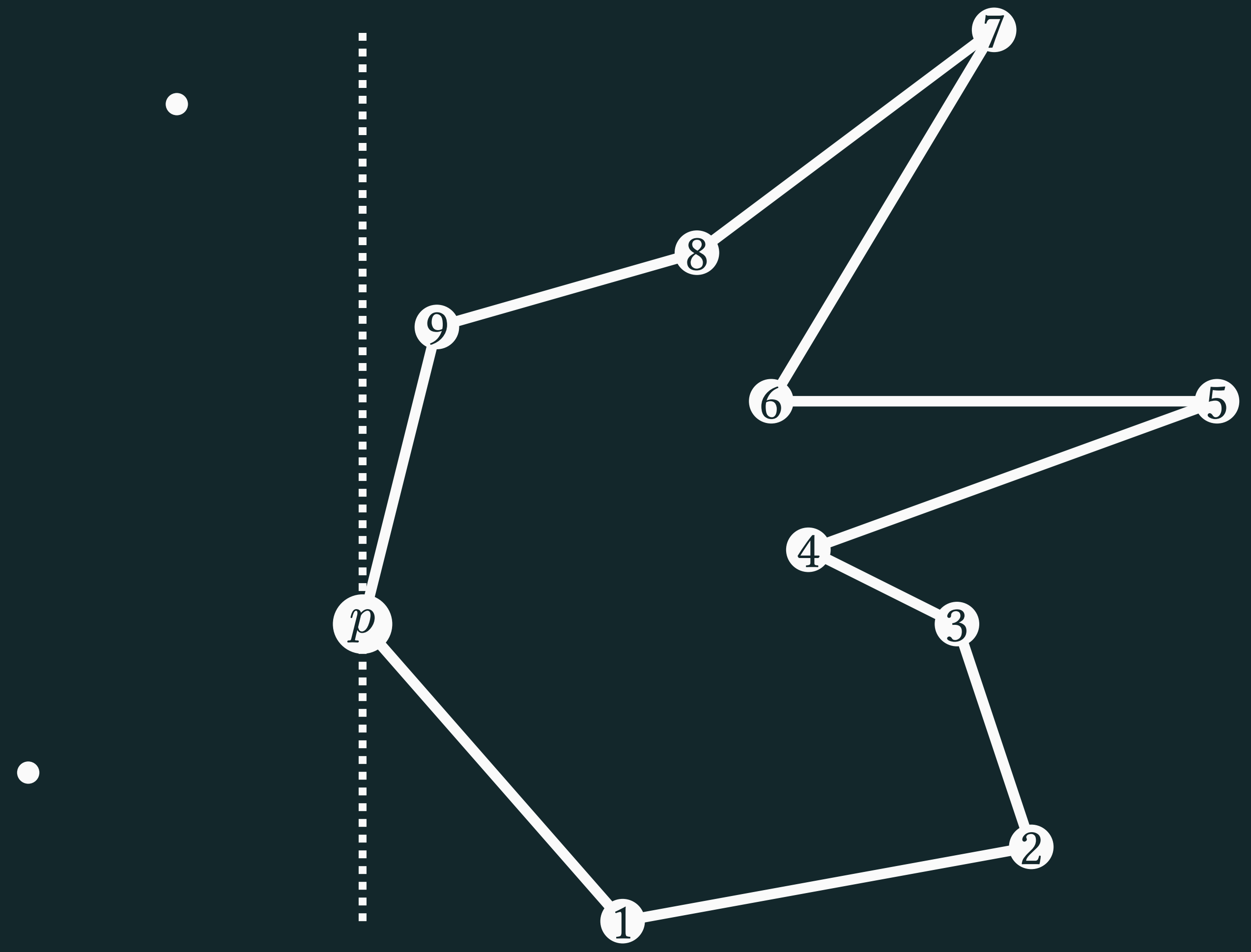
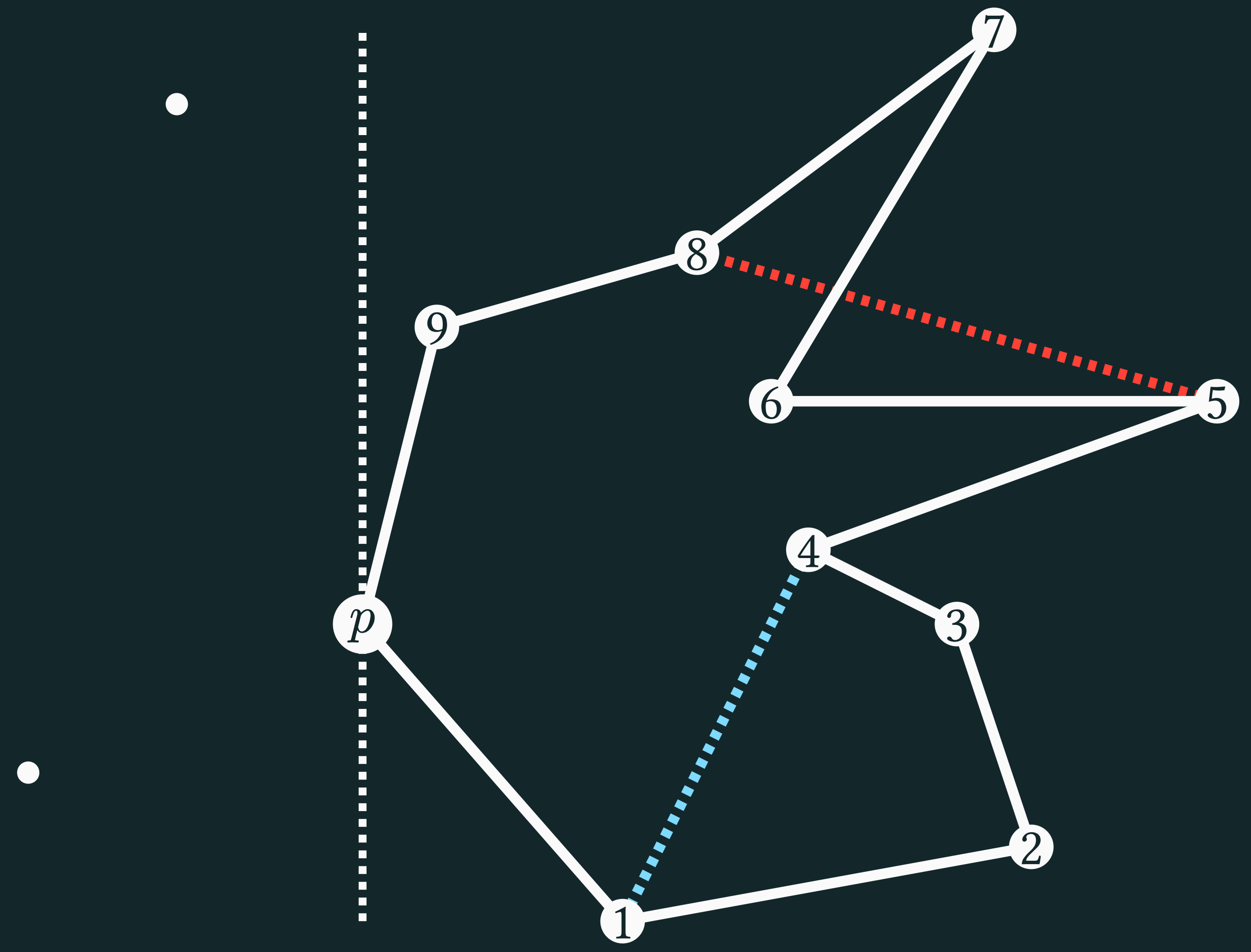


Hole-finding algorithm



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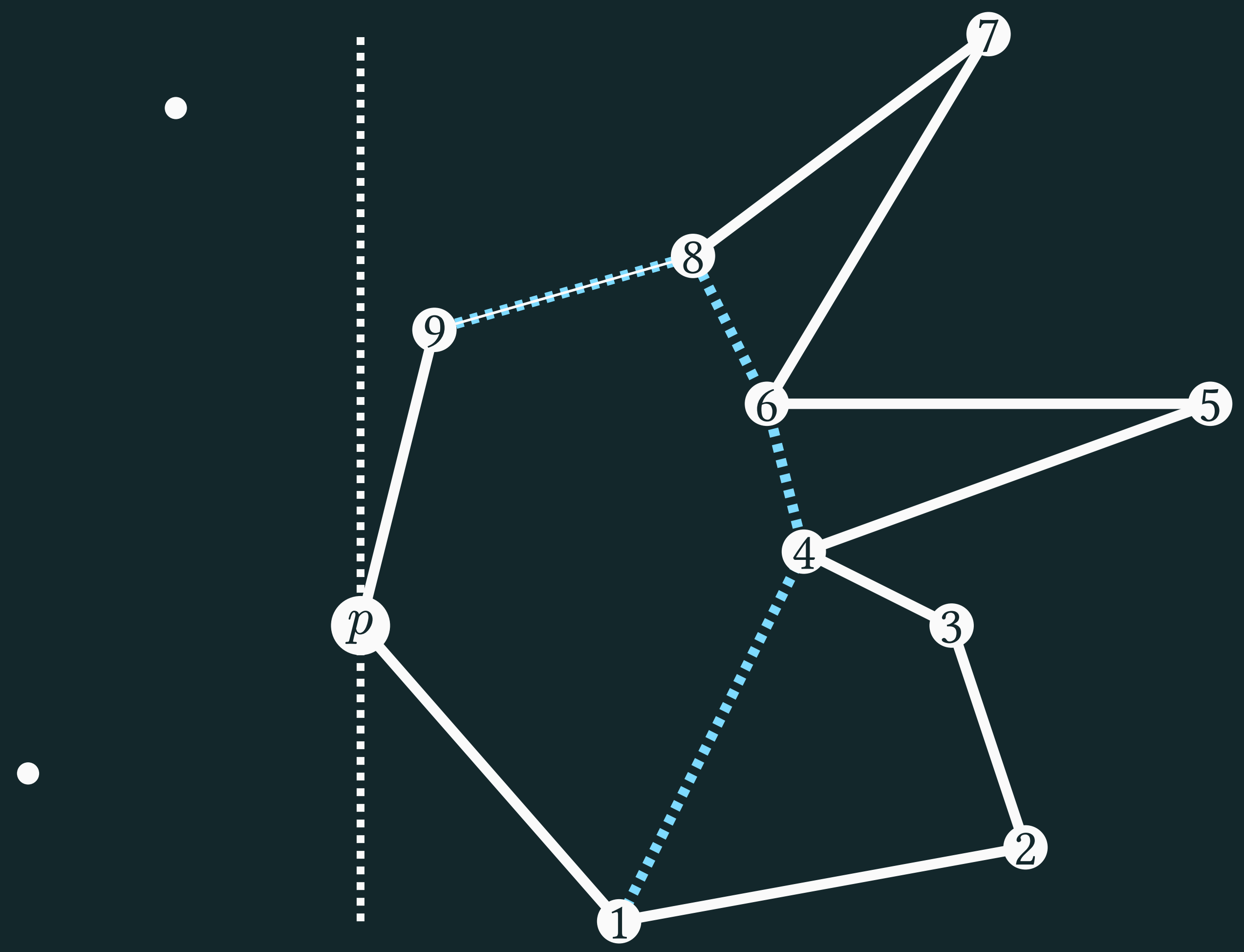

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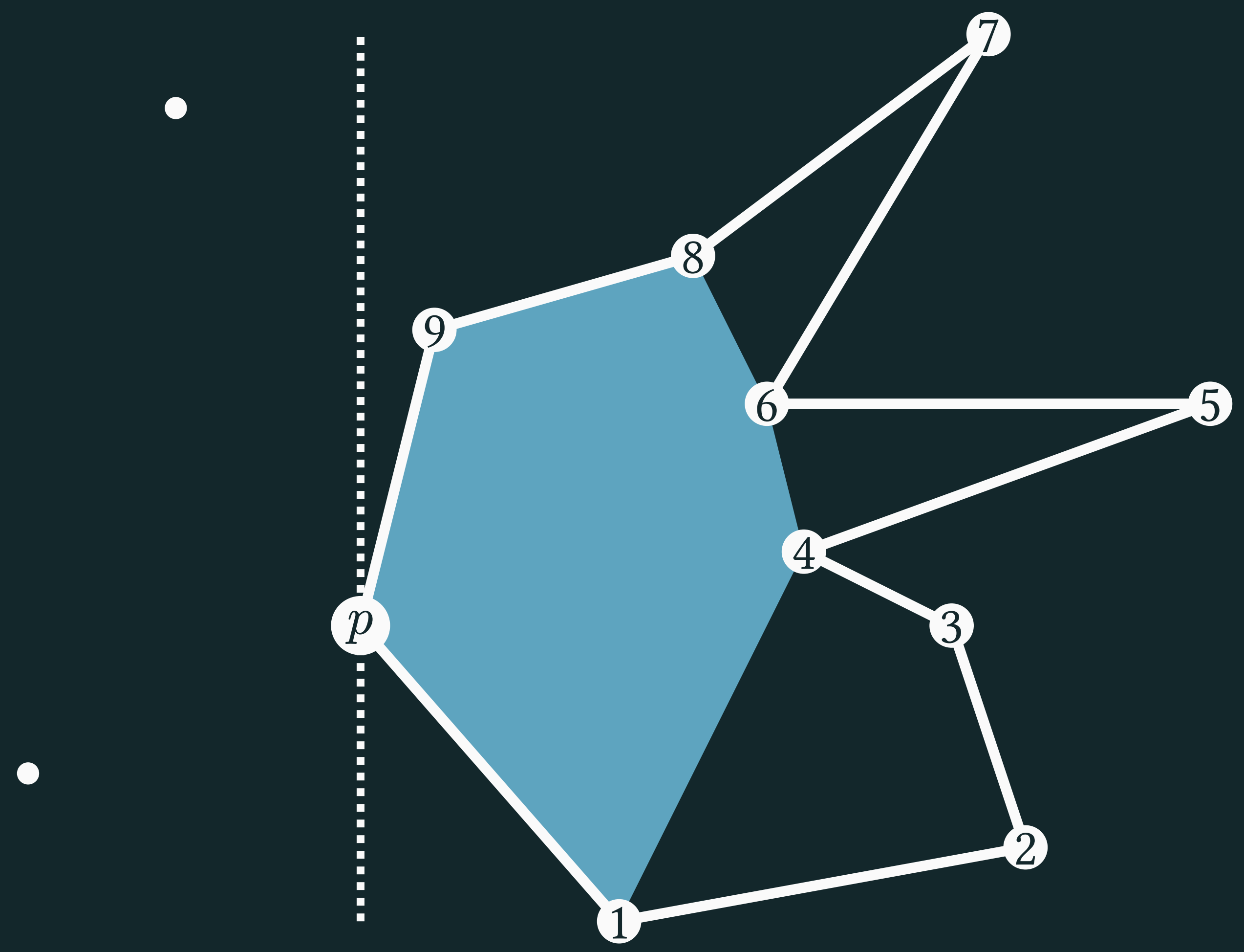

Hole-finding algorithm: verification

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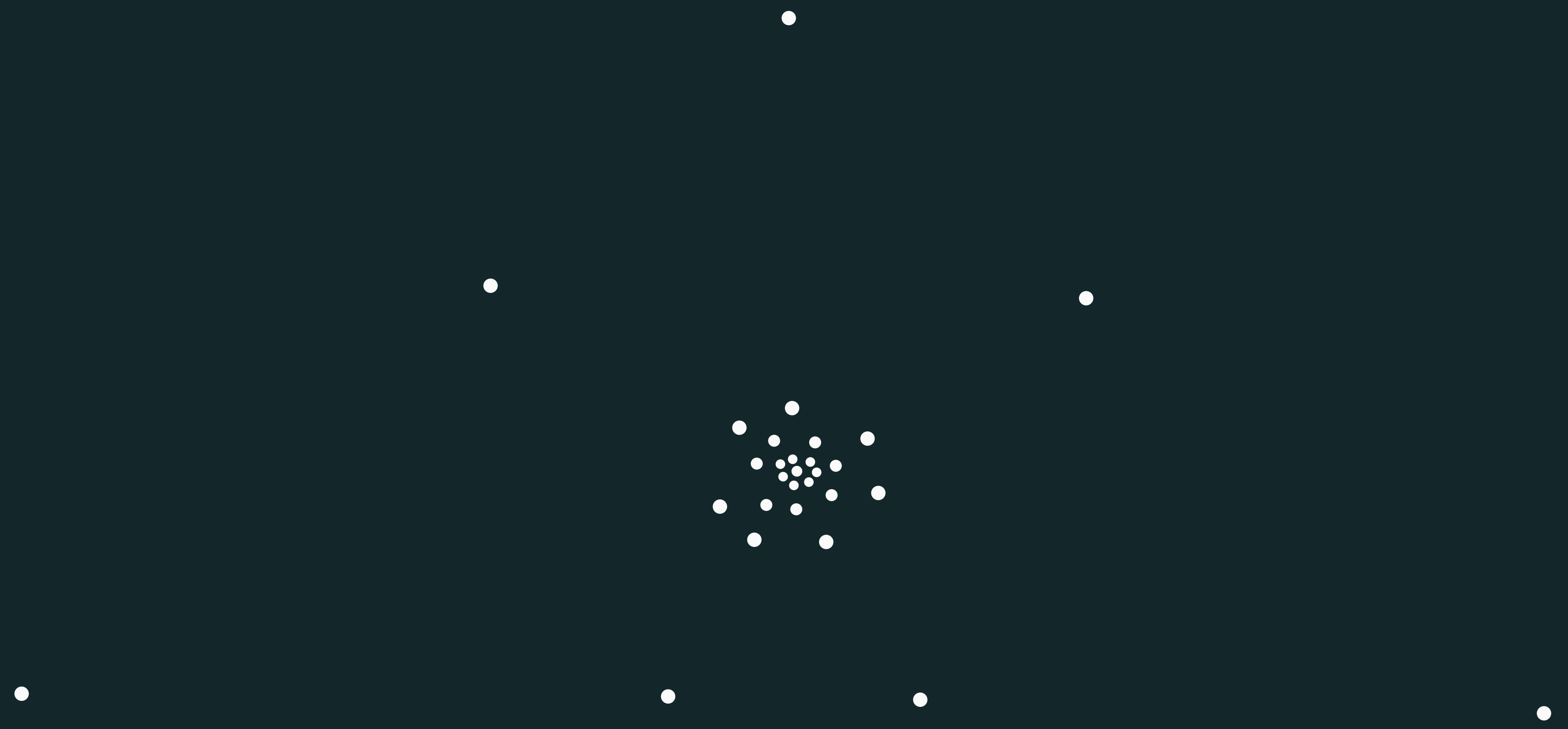
```
theorem of_proceed_loop
    {i j : Fin n} (ij : Visible p pts i j) {Q : Queues n j} {Q_j : BelowList n j} {Q_i} (ha)
    {IH} (hIH : ∀ a (ha : a < i), Visible p pts a j → ProceedIH p pts (ha.trans ij.1) (IH a ha))
    (Hj : Queues.OrderedTail p pts lo j (fun k h => Q.q[k.1]'(Q.sz ▸ h)) Q_j.1)
    (ord : Queues.Ordered p pts lo i (fun k h => Q.q[k.1]'(Q.sz ▸ h.trans ij.1)) Q_i)
    (g_wf : Q.graph.WF (VisibleLT p pts lo j))
    {Q' Q_j'} (eq : proceed.loop pts i j ij.1 IH Q Q_j Q_i ha = (Q', Q_j')) :
    ∃ a Q₁ Q_i₁ Q_j₁, proceed.finish i j ij.1 Q₁ Q_i₁ Q_j₁ = (Q', Q_j') ∧
      Q₁.graph.WF (VisibleLT p pts i j) ∧
      (∀ k ∈ Q_i₁.1, σ (pts k) (pts i) (pts j) ≠ .ccw) ∧
      lo ≤ a ∧ Queues.Ordered p pts a i (fun k h => Q.q[k.1]'(Q.sz ▸ h.trans ij.1)) Q_i₁.1 ∧
      (∀ (k : Fin n) (h : k < j), ¬(lo ≤ k ∧ k < a) → Q₁.q[k.1]'(Q₁.sz ▸ h) = Q.q[k.1]'(Q.sz ▸ h)) ∧
      Queues.OrderedTail p pts a j (fun k h => Q₁.q[k.1]'(Q₁.sz ▸ h)) Q_j₁.1 := by
```

Lower bound: 29 points with no 6-holes (Overmars 2003)



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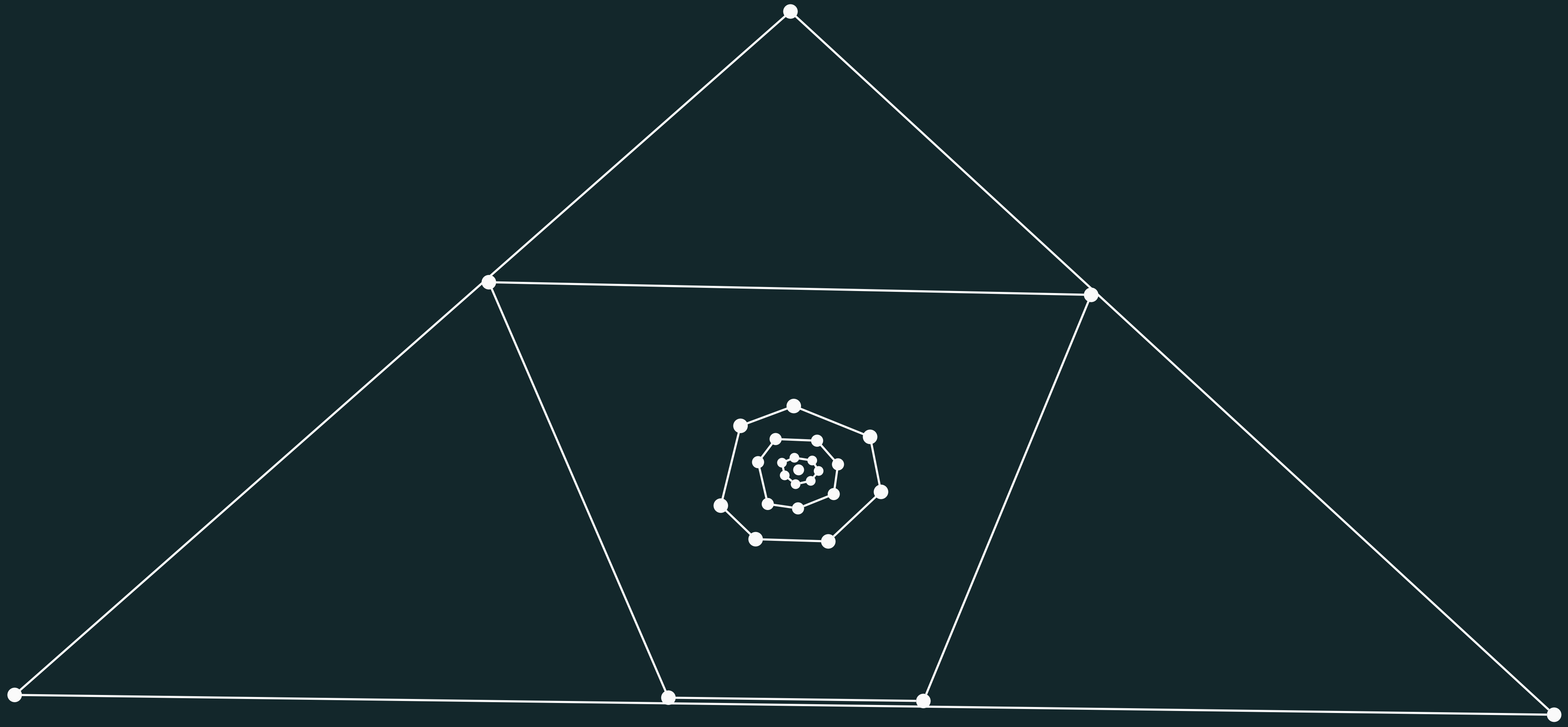

Final theorem

```
axiom unsat_6hole_cnf : (Geo.hexagonCNF 30).isUnsat

theorem holeNumber_6 : holeNumber 6 = 30 :=
  le_antisymm
   (hole_6_theorem' unsat_6hole_cnf)
   (hole_lower_bound [
    (1, 1260),  (16, 743),  (22, 531),  (37, 0),    (306, 592),
    (310, 531), (366, 552), (371, 487), (374, 525), (392, 575),
    (396, 613), (410, 539), (416, 550), (426, 526), (434, 552),
    (436, 535), (446, 565), (449, 518), (450, 498), (453, 542),
    (458, 526), (489, 537), (492, 502), (496, 579), (516, 467),
    (552, 502), (754, 697), (777, 194), (1259, 320)])
```

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- Trust story for large SAT proofs could be improved.

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